Cox rings of minimal resolutions of surface quotient singularities

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Abstract. We investigate Cox rings of minimal resolutions of surface quotient singularities and provide two descriptions of these rings. The first one is the equation for the spectrum of a Cox ring, which is a hypersurface in an affine space. The second is the set of generators of the Cox ring viewed as a subring of the coordinate ring of a product of a torus and another surface quotient singularity. In addition, we obtain an explicit description of the minimal resolution as a divisor in a toric variety.

1. Introduction

Let X be a normal (pre)variety with free finitely generated class group. The Cox ring (or the total coordinate ring) of X is a $\mathrm{Cl}(X)$-graded module

$$\mathrm{Cox}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D))$$

with multiplication as in the field of rational functions on X. Different choices of representatives of divisor classes give isomorphic ring structures. A very important question is whether the Cox ring of a variety is finitely generated.

One can describe this object from a geometric point of view if only $\mathrm{Cox}(X)$ is finitely generated. Assume $\mathrm{Pic}(X)$ is torsion-free and consider the action of the Picard torus of X

$$T = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{C}^*)$$

on $\mathrm{Spec}(\mathrm{Cox}(X))$. Then X can be obtained as a geometric quotient of an open subset of $\mathrm{Spec}(\mathrm{Cox}(X))$ by T. Thus the Cox ring contains a lot of information on the geometry of X – the variety is determined by $\mathrm{Spec}(\mathrm{Cox}(X))$ and some combinatorial data in its grading group (see e.g. [**LV09**]).

In this paper we study the case of minimal resolutions of surface quotient singularities (over $\mathbb{C}$), i.e. X is a minimal resolution of the quotient space $\mathbb{C}^2/G$ for a finite subgroup $G \subset GL(2,\mathbb{C})$. In this case $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ is torsion-free (see Proposition 2.10).

The main result is a description of the Cox rings of minimal resolutions of surface quotient singularities: in terms of a single equation for its spectrum (Theorem 5.3),

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and also by a (finite) set of generators, as a subring of the coordinate ring of the product of the Picard torus and a singular surface (Theorem 6.9). While the first of these theorems is related to the results of [**HS10**] and can be proven using the ideas presented there, the second one introduces a new method of describing the Cox ring, designed to work in the case of quotient singularities. Thus the present paper can be thought of as a first step towards understanding the total coordinate rings of resolutions of quotient singularities in general. The ideas described in its final part possibly can be generalized and applied to higher-dimensional singularities. As we attempt to develop methods that could work also in more complex cases, we do everything step by step, performing quite a lot of computations, checking details and providing examples.

An important motivation for extending this work is the possibility of presenting X as a geometric quotient of an open set of $\mathrm{Spec}(\mathrm{Cox}(X))$ in case where $\mathrm{Cox}(X)$ is finitely generated. Roughly speaking, if one finds a way to understand the Cox ring of a (hypothetical) resolution X of a (quotient) singularity, based only on some restricted knowledge of the geometry of X, one may be able to construct some new resolutions as geometric quotients of open sets of $\mathrm{Spec}(\mathrm{Cox}(X))$. An especially interesting case is the one of 4-dimensional symplectic quotient singularities and their symplectic resolutions. The potential results may be used in the work on a generalization of the classical Kummer construction, investigated in [**AW10**] and [**Don11**].

The first attempt to study Cox rings of resolutions of quotient singularities is a recent paper [**FGAL11**], where the authors find the single relation of the $\mathrm{Cox}(X)$ where X is the minimal resolution of a Du Val singularity (i.e. $G \subset SL(2, \mathbb{C})$). However, their methods rely heavily on the equations of an embedding of the singularity in an affine space, and consequently their work seems to be very hard to generalize in a straightforward manner. Cox rings of minimal resolutions of all surface quotient singularities can be computed using the theory of varieties endowed with a (diagonal) torus action such that its biggest orbits are of codimension one, see [**HS10**]. However, these results also do not apply to singularities in higher dimensions.

1.1. Outline of the paper. Throughout the paper X denotes the minimal resolution of a surface quotient singularity $\mathbb{C}^2/G$, where G is a finite subgroup of $GL(2, \mathbb{C})$.

In section 2 we recall basic informations on the set-up: properties of finite (small) subgroups of $GL(2, \mathbb{C})$ and the structure of the special fibre of their minimal resolution (after [**Bri68**]). Then, in section 3, we define an action of the Picard torus T of X on an affine space which will become the ambient space for $\mathrm{Spec}(\mathrm{Cox}(X))$. We investigate the properties of this action in the toric setting and describe the quotient as a toric variety. In section 3.3 a candidate S for $\mathrm{Spec}(\mathrm{Cox}(X))$ is proposed. It is defined as a T-invariant hypersurface in an affine space. In section 4 we describe a geometric quotient of an open subset of S by the action of T as a divisor in a toric variety and prove that it is the minimal resolution of $\mathbb{C}^2/G$. This may seem to be a roundabout way of reproving the results of Brieskorn [**Bri68**]. However, we are planning to use the ideas developed in this work in cases of higher dimensional quotient singularities, where the resolutions do not have such a detailed description, and try to reverse the process: construct resolutions of quotient singularities from their Cox rings.

Section 5 contains the proof of the first of the main results, Theorem 5.3, which states that S is the spectrum of the Cox ring of X. The proof is based on [**ADHL10**, Thm. 6.4.3], the GIT characterization of the Cox ring. The last section contains the second very important result, Theorem 6.9. It is a description of $\mathrm{Cox}(X)$ in terms of its generators, as a subring of $\mathbb{C}[x,y]^{[G,G]} \otimes \mathbb{C}[t_0^{\pm 1}, \ldots, t_{n-1}^{\pm 1}]$, where $[G,G]$ denotes the commutator subgroup of G. We hope that the last part of the paper will be the basis for generalizing these results to higher dimensions.

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Also, thanks to Michał Lasoń and Víctor González Alonso for answering the questions regarding their work [**FGAL11**].

2. The background material

This section contains the list of groups for which we consider the quotient singularity $\mathbb{C}^2/G$. We describe the minimal resolution of these singularities. We also compute commutator subgroups and abelianizations of considered groups, which will be needed in the sequel, especially in section 6.

2.1. Groups. We investigate the singularities constructed by taking the quotient of $\mathbb{C}^2$ by the linear action of a finite subgroup of $GL(2,\mathbb{C})$. Such a quotient either is smooth or has an isolated singularity in 0. However, it is worth noting that in higher dimensions the singular locus of a quotient of an affine space by a finite linear group action can be much more complicated.
The Chevalley-Shephard-Todd theorem states that the ring of invariants of such a group action is a polynomial ring if and only if the group is generated by pseudo-reflections (a proof can be found in [**Stu93**, Section 2.4]). Due to this result we can restrict ourselves to considering small groups, i.e. groups without pseudo-reflections. A pseudo-reflection is a linear transformation of dimension n which has 1 as an eigenvalue with multiplicity $n-1$. Finite small subgroups of $GL(2,\mathbb{C})$ are classified and listed e.g. in [**Bri68**, Satz 2.9] and in [**Rie77**]. It is worth noting that the conjugacy classes of the non-cyclic small subgroups of $GL(2,\mathbb{C})$ coincide with their isomorphism classes. Before listing the groups we recall the notation for the fibre product cases, repeated after [**Bri68**].

Notation 2.1. Let $\mu : GL(2,\mathbb{C}) \times GL(2,\mathbb{C}) \to GL(2,\mathbb{C})$ be the matrix multiplication. Take $H_1, H_2 \subset GL(2,\mathbb{C})$ with normal subgroups N_1 and N_2 respectively, such that there is an isomorphism $\phi : H_1/N_1 \to H_2/N_2$. By $[h_i]$ we denote the class of $h_i \in H_i$ in H_i/N_i. We will consider the image under μ of the fibre product of H_1 and H_2 over ϕ:

$$(H_1, N_1; H_2, N_2)_\phi = \mu(\{(h_1, h_2) \in H_1 \times H_2 \;:\; [h_2] = \phi([h_1])\}).$$

If the choice of ϕ is obvious, it will be denoted by $(H_1, N_1; H_2, N_2)$.

The conjugacy classes of finite small subgroups of $GL(2,\mathbb{C})$ are:

(1) cyclic groups $C_{n,q} = diag(\varepsilon_n, \varepsilon_n^q)$, where as usually $\varepsilon_n = e^{2\pi i/n}$,
(2) non-cyclic groups contained in $SL(2,\mathbb{C})$:

- binary dihedral groups BD_n ($4n$ elements, $n \geq 2$, gives the Du Val singularity D_{n+2}),
- binary tetrahedral group BT (24 elements, the singularity is E_6),
- binary octahedral group BO (48 elements, the singularity is E_7),
- binary icosahedral group BI (120 elements, the singularity is E_8),

(3) images under μ of fibre products of a group in $SL(2,\mathbb{C})$ and a cyclic group $Z_k = C_{k,k-1} = diag(\varepsilon_k, \varepsilon_k^{k-1})$:
  - $BD_{n,m}$ for $(m,n) = 1$, defined as $(Z_{2m}, Z_{2m}; BD_n, BD_n)$ for odd m and $(Z_{4m}, Z_{2m}; BD_n, C_{2n})$, where $C_{2n} \lhd BD_n$ is cyclic of order $2n$, when m is even,
  - BT_m defined as $(Z_{2m}, Z_{2m}; BT, BT)$ in the cases where $(m,6) = 1$ and as $(Z_{6m}, Z_{2m}; BT, BD_2)$ when $(m,6) = 3$,
  - $BO_m = (Z_{2m}, Z_{2m}; BO, BO)$ if $(m,6) = 1$,
  - $BI_m = (Z_{2m}, Z_{2m}; BI, BI)$ if $(m,30) = 1$.

  Note that for $m = 1$ then we obtain the subgroups of $SL(2,\mathbb{C})$ listed above.

Generators of each of these groups can be found in [**Rie77**].
Quotients by cyclic groups are toric singularities and thus their Cox rings are polynomial rings of $n+1$ variables. Hence we will not describe this case; the details can be found e.g. in [**CLS11**, Chapter 5].
The generators of the subgroups of $SL(2,\mathbb{C})$ are given in [**Rei**]. A simple computation (for example in [**GAP12**]) shows that the commutator subgroups and the abelianizations are:

- $[BD_n, BD_n] \simeq \mathbb{Z}_n$, it is generated by $diag(\varepsilon_n, \varepsilon_n^{-1})$, $Ab(BD_n)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even and $\mathbb{Z}_4$ for odd n,
- $[BT, BT] = BD_2$, $Ab(BT) \simeq \mathbb{Z}_3$,
- $[BO, BO] = BT$, $Ab(BO) \simeq \mathbb{Z}_2$,
- $[BI, BI] = BI$, $Ab(BI) = 1$.

Lemma 2.2. *The commutator subgroups of the small subgroups of $GL(2,\mathbb{C})$ not contained in $SL(2,\mathbb{C})$ are the same as the commutator subgroups of the non-cyclic factors of the corresponding fibre product structure.*

Proof. Consider $G = (H_1, N_1; H_2, N_2)$ such that H_1 is in the center of $GL(2,\mathbb{C})$. Take $g, g' \in G$, let $g = h_1 h_2$ and $g' = h_1' h_2'$ where $h_i, h_i' \in H_i$. Then $gg'g^{-1}g'^{-1} = h_2 h_2' h_2^{-1} h_2'^{-1}$, so $[G,G] \subseteq [H_2, H_2]$. They are equal, as every element of H_2 appears in G multiplied by some element of H_1. □

The abelianizations of considered groups are now easy to compute. Their isomorphism types are given in [**Bri68**, p. 348] in the last column of the table. However, in the proof of Proposition 4.6 we need to know the generators of $Ab(G)$ (written as matrices in $GL(2,\mathbb{C})$ whose classes generate $G/[G,G]$).
Let $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $C = \frac{1}{2}\begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$.

Corollary 2.3. *The abelianizations of considered groups are*

- *if n is even $Ab(BD_{n,m}) \simeq \mathbb{Z}_{2m} \times \mathbb{Z}_2$ is generated by $\varepsilon_{2m} \cdot B$ and $diag(\varepsilon_{2n}, \varepsilon_{2n}^{-1})$,*
- *if n is odd $Ab(BD_{n,m}) \simeq \mathbb{Z}_{4m}$ is generated by $\varepsilon_{4m} \cdot B$,*
- *$Ab(BT_m) \simeq \mathbb{Z}_{3m}$ is generated by*
  - *$\varepsilon_{2m} \cdot C$ if $(m,6) = 1$,*

– $\varepsilon_{6m} \cdot C$ *if* $(m, 6) = 3$,
- $Ab(BO_m) \simeq \mathbb{Z}_{2m}$ *is generated by* $\varepsilon_{2m} \cdot diag(\varepsilon_8, \varepsilon_8^{-1})$,
- $Ab(BI_m) \simeq \mathbb{Z}_m$ *is generated by* $diag(\varepsilon_m, \varepsilon_m)$.

PROOF. The order of $Ab(G)$ can be easily computed – we know the order of $[G, G]$ and we only have to determine the order of the kernel of μ on the fibre product of a cyclic group generated by $diag(\varepsilon_{2k}, \varepsilon_{2k})$ and a subgroup of $SL(2, \mathbb{C})$. And it is always $\mathbb{Z}_2$, because if $0 < i \leq 2k$ and $M \in SL(2, \mathbb{C})$ is nontrivial then $\varepsilon_{2k}^i \cdot M = 1$ if and only if $i = k$ and $M = diag(-1, -1)$; moreover, this pair of matrices is always in the fibre product.
Hence in all the cases of cyclic abelianizations, i.e. all but the first one, it suffices to say that the order of the element given in the formulation of the corollary modulo $[G, G]$ is in fact equal to the order of $Ab(G)$. In the case of $G = BD_{n,m}$ for n even, $Ab(G)$ has $4m$ elements, $\varepsilon_{2m} \cdot B$ is of order $2m$ and $diag(\varepsilon_{2n}, \varepsilon_{2n}^{-1})$ of order 2 modulo $[G, G]$. The classes of these elements in $Ab(G)$ commute. Moreover, $(\varepsilon_{2m} \cdot B)^m = -B^m$ and $diag(\varepsilon_{2n}, \varepsilon_{2n}^{-1})$ represent different elements of $Ab(G)$, because $[G, G] = \langle (diag(\varepsilon_{2n}, \varepsilon_{2n}^{-1}))^2 \rangle$. Thus in fact $Ab(G) \simeq \mathbb{Z}_{2m} \times \mathbb{Z}_2$. $\square$

2.2. Resolution of singularities. The singularities we consider are rational, so the exceptional divisor of the minimal resolution is a tree of smooth rational curves, meeting transversally. More accurately, for the surface quotient singularities it is a tree with three chains of curves starting from a central curve. In other words, the dual graph is a T-shaped diagram (see Fig. 1). For the subgroups of $SL(2, \mathbb{C})$ they are Dynkin diagrams of the root systems D_n for $n \geq 4$, E_6, E_7 and E_8. In this case all rational curves in the exceptional fibre have self-intersection -2. For the groups not contained in $SL(2, \mathbb{C})$ the diagrams do not have to be Dynkin diagrams any more, and also the self-intersection numbers can be less then -2, as it is shown in Examples 2.5 and 2.6. The structure of the exceptional fibres for small subgroups of $GL(2, \mathbb{C})$ is described in details for example in [**Bri68**] and [**Rie77**]. Here we recall some facts which will be useful in the further part of this text. First of all we fix some notation.
For a chosen small group $G \subset GL(2, \mathbb{C})$ we will denote by X the minimal resolution of the quotient singularity $\mathbb{C}^2/G$ (it is unique, as we consider only the surface case). We describe the special fibre of the resolution $X \to \mathbb{C}^2/G$.

NOTATION 2.4. Let E_0 be the curve corresponding to the branching point of the diagram and $E_{i,j}$ be the j-th curve in the i-th branch, counting from E_0, as in Fig. 1. We assume that the first branch always has the smallest length.

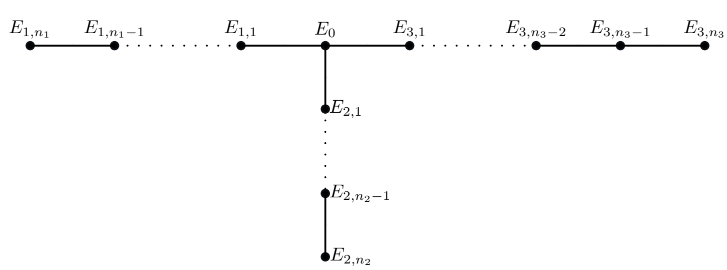


FIGURE 1. Dual graph of the special fibre of the minimal resolution of $\mathbb{C}^2/G$

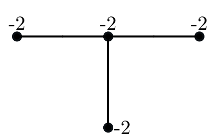
If we need to write the curves in the special fibre in a sequence, we order them as follows:

$$E_0; E_{1,1}, \ldots, E_{1,n_1}; E_{2,1}, \ldots, E_{2,n_2}; E_{3,1}, \ldots, E_{3,n_3}.$$

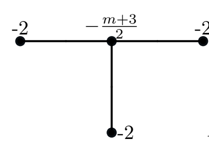
The number of irreducible components of the special fibre is $n = n_1 + n_2 + n_3 + 1$.

Full information about the intersection numbers of all curves in the special fibre is contained in an invariant consisting of seven numbers: $\langle d; p_1, q_1; p_2, q_2; p_3, q_3\rangle$, which we associate to the structure of a non-cyclic small group $G \subset GL(2, \mathbb{C})$. Broadly speaking, these numbers are connected to the fibre product description of the group structure (see section 2.1). This follows from the construction of the resolution from the well-understood minimal resolutions for the subgroups of $SL(2, \mathbb{C})$ – for the details we refer the reader to [**Bri68**]. Here we only need the fact that $d = -E_0.E_0$, and the remaining six numbers divide into three pairs which, expanded into the Hirzebruch-Jung continued fractions (defined e.g. [**Rei97**]), give sequences of the negatives of the intersection numbers for the curves in three branches. The exact rule to restore these numbers from the group structure, used in the examples below, is [**Bri68**, Satz 2.11].

Example 2.5. Quotients by $BD_{2,m}$. The simplest case is $BD_{2,1} = BD_2 \subset SL(2, \mathbb{C})$, which gives the Du Val singularity D_4 whose dual graph has three branches of length 1. As $(m, n) = 1$, the other cases are $(\mathbb{Z}_{2m}, \mathbb{Z}_{2m}; BD_2, BD_2)$ for m odd. It turns out that the dual graphs of the resolutions are the same as for BD_2, but the self-intersection number in the branching point changes: for $BD_{2,m}$ it is $-\frac{m+3}{2}$. Using the notation of [**Bri68**], the resolution is described by the sequence $\langle -\frac{m+3}{2}; 2, 1; 2, 1; 2, 1\rangle$.



BD_2

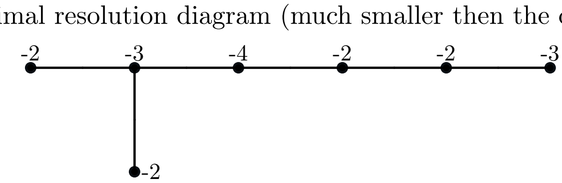


$BD_{2,m}$

Example 2.6. Starting from larger binary dihedral groups and taking the fibre product with a suitable subgroup of the center one can obtain resolutions much different from the Du Val case. For example, for $BD_{23,39}$ the resolution is described by the sequence $\langle d; 2, 1; 2, 1; 23, q\rangle$, where $39 = 23(d-1) - q$. Thus $d = 3$ and $q = 7$, the Hirzebruch-Jung describing the last branch is

$$\frac{23}{7} = 4 - \frac{1}{2 - \frac{1}{2 - \frac{1}{3}}}$$

and the minimal resolution diagram (much smaller then the one for BD_{23}) is



$BD_{23,39}$

Based on the intersection numbers of curves in the special fibre of the resolution we define a matrix U which will be called an *extended intersection matrix* for the singularity $\mathbb{C}^2/G$.

We start from the intersection matrix U^0 of the rational curves in the special fibre of the minimal resolution. The curves are ordered as stated in 2.4. Thus $U^0_{k,l}$ is

the intersection number of the k-th and l-th curve in the sequence. We extend U^0 to a matrix U by adding three columns. One can imagine that to the ending curve of each branch we add a leaf – a curve which intersect (transversally) the last curve in a branch, but which is not an element of the exceptional fibre, so we do not include its self-intersection number. Hence, for $i = 1, 2, 3$, just after the column corresponding to E_{i,n_i} we add a column filled with 0 except of the entry corresponding to E_{i,n_i}, where we put 1. This construction will be used to define an action of a torus on the (candidate for the) spectrum of the Cox ring of X, introduced in section 3.

Notation 2.7. Throughout the paper we will think of U as if it was divided in a few blocks:

$$\left(\begin{array}{c|c|c|c|c|c|c}
-d & 1\ \ 0\ \ \dots\ \ 0 & 0 & 1\ \ 0\ \ \dots\ \ 0 & 0 & 1\ \ 0\ \ \dots\ \ 0 & 0 \\ \hline
\begin{matrix}1\\0\\\vdots\\0\end{matrix} & A_1 & \begin{matrix}0\\\vdots\\0\\1\end{matrix} & 0 & 0 & 0 & 0 \\ \hline
\begin{matrix}1\\0\\\vdots\\0\end{matrix} & 0 & 0 & A_2 & \begin{matrix}0\\\vdots\\0\\1\end{matrix} & 0 & 0 \\ \hline
\begin{matrix}1\\0\\\vdots\\0\end{matrix} & 0 & 0 & 0 & 0 & A_3 & \begin{matrix}0\\\vdots\\0\\1\end{matrix}
\end{array}\right)$$

Here A_i is the matrix of intersection numbers of curves in the i-th branch of the diagram of the minimal resolution:

$$A_i = \begin{pmatrix}
-a_{i,1} & 1 & 0 & 0 & 0 & 0 \\
1 & -a_{i,2} & 1 & 0 & 0 & 0 \\
0 & 1 & -a_{i,3} & 1 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & 1 & -a_{i,n_i-1} & 1 \\
0 & 0 & 0 & 0 & 1 & -a_{i,n_i}
\end{pmatrix}$$

On the diagonal there is the sequence of the negatives of elements of a Hirzebruch-Jung continued fraction associated to the i-th branch. This means that A_i is just the intersection matrix of the components of the exceptional divisor of a certain cyclic quotient singularity (see [**Rei97**] and [**Bri68**]). This singularity will appear later in the toric geometry picture of the considered situation in section 3.2.

In fact, the results of Brieskorn give more restrictions for the description of the special fibre of X. In particular, not all T-shaped diagrams can appear as dual graphs of the special fibre. It turns out that one branch always has length one and also the second cannot be too long. Moreover, there are many restriction for the self-intersection numbers. Here is a more detailed description.

Remark 2.8. According to [**Bri68**, p. 348],

- $A_1 = (-2)$,
- at least one of A_2, A_3 is one of (-2), $\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, (-3),
- A_3 can be of any size only if $A_1 = A_2 = (-2)$; otherwise A_3 is at most 5×5 matrix.

Finally, we describe the Picard group and the class group of the singularity and the resolution.

Proposition 2.9. *For the singularity $\mathbb{C}^2/G$ we have*

$$\mathrm{Pic}(\mathbb{C}^2/G) = 0, \qquad \mathrm{Cl}(\mathbb{C}^2/G) \simeq Ab(G).$$

Proof. These two properties are Theorems 3.6.1 and 3.9.2 in [**Ben93**]. $\square$

Proposition 2.10. *If n is the number of exceptional curves in the minimal resolution X of $\mathbb{C}^2/G$, then*

$$\mathrm{Pic}(X) = \mathrm{Cl}(X) \simeq \mathbb{Z}^n.$$

Proof. We start from showing that $\mathrm{Cl}(X) = \mathrm{Pic}(X)$ is a lattice. First notice that $\pi_1(X)$ is trivial, as $\mathbb{C}^2/G$ is contractible and by [**Kol93**, Thm. 7.8] the blow-ups do not change the fundamental group. Hence $H_1(X, \mathbb{Z}) = 0$ and from the exponential sequence we deduce that $\mathrm{Pic}(X) \subseteq H^2(X, \mathbb{Z})$. By the universal coefficient theorem we have an exact sequence

$$0 \to \mathrm{Ext}(H_1(X, \mathbb{Z}), \mathbb{Z}) \to H^2(X, \mathbb{Z}) \to \mathrm{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z}) \to 0,$$

in which the first term is 0 and the last one is a free abelian group, so $H^2(X, \mathbb{Z})$ is also torsion-free. Thus $\mathrm{Pic}(X)$ is indeed a lattice and we just have to show that its rank is the number of irreducible components of the special fibre of the resolution. Then there is an exact sequence

$$0 \to \mathbb{Z}^n \to \mathrm{Cl}(X) \to \mathrm{Cl}(\mathbb{C}^2/G) \simeq Ab(G) \to 0,$$

where the first term is the lattice spanned by the exceptional curves, which are exactly the divisors contracted by the resolution morphism $X \to \mathbb{C}^2/G$. By Corollary 2.3 $Ab(G)$ is a finite group, so the middle term must be $\mathbb{Z}^n$. $\square$

3. The Picard torus action

Let n be the number of components of the exceptional fibre of the minimal resolution X of $\mathbb{C}^2/G$ for a small subgroup $G < GL(2, \mathbb{C})$. We define the action of the Picard torus $T = \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{C}^*) \simeq (\mathbb{C}^*)^n$ on $\mathbb{C}^{n+3}$ and investigate geometric quotients of open subsets of this affine space. Then, in section 3.3, we propose a candidate for the $\mathrm{Spec}(\mathrm{Cox}(X))$, defined as a hypersurface in $\mathbb{C}^{n+3}$, and prove that it is invariant under the action of T in order to consider its quotients by T (see section 4). An inspiration for this part of the paper is the construction of the total coordinate ring of a toric variety described in [**CLS11**, Section 5.2].
To define the action of T on $\mathbb{C}^{n+3}$ we use the extended intersection matrix U, described in Notation 2.7. We fix the coordinates: let

$$\mathbb{C}[y_0, y_{1,1}, \ldots, y_{1,n_1}, x_1, y_{2,1}, \ldots, y_{2,n_2}, x_2, y_{3,1}, \ldots, y_{3,n_3}, x_3]$$

be the coordinate ring of $\mathbb{C}^{n+3}$. Then we define the map

$$\varphi : T \times \mathbb{C}^{n+3} \to \mathbb{C}^{n+3}$$

with the formula

$$(3.0.1)\quad (\underline{t},\underline{x}) = ((t_1,\dots,t_n),(y_0,y_{1,1},\dots,y_{3,n_3},x_3)) \mapsto \\ \mapsto (\underline{t}^{u_0}\cdot y_0, \underline{t}^{u_1}\cdot y_{1,1},\dots,\underline{t}^{u_{n-1}}\cdot y_{3,n_3}, \underline{t}^{u_n}\cdot x_3)$$

where u_i is the i-th column of U and $\underline{t}^{u_i} = t_1^{(u_i)_1}\cdots t_n^{(u_i)_n}$.
In section 3.2 we consider certain geometric quotients of open subsets of $\mathbb{C}^{n+3}$ by this action of T. Toric geometry is used there to give a detailed description of the quotients. But first we need some technical results: in section 3.1 we determine the kernel of U (viewed as a map of lattices of integral points in vector spaces over $\mathbb{R}$), which appears later in the toric setting.

3.1. The kernel map. We look at U as at the restriction of a map from $\mathbb{R}^{n+3}$ to $\mathbb{R}^n$ (in the standard basis) to the sublattice $\mathbb{Z}^{n+3}\subset\mathbb{R}^{n+3}$. By $\ker U$ we understand the sublattice of $\mathbb{Z}^{n+3}$ carried to 0 by U. The aim of this section is to describe a convenient set of its generators. To begin with, we need to define some matrix modifications which will be applied to the matrices A_i describing branches of the resolution diagram.

DEFINITION 3.1. Let A be a square matrix. Then A' denotes A with a row $(1,0,\dots,0)$ added at the top and a column $(0,\dots,0,1)$ added on the right:

$$A' = \left(\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ \hline & & & & 0 \\ & & & & \vdots \\ & & A & & \\ & & & & 0 \\ & & & & 1 \end{array}\right)$$

A'' denotes A with the columns $(1,0,\dots,0)$ and $(0,\dots,0,1)$ added on the left and on the right respectively:

$$A'' = \left(\begin{array}{c|c|c} 1 & & 0 \\ 0 & & \vdots \\ & A & \\ \vdots & & 0 \\ 0 & & 1 \end{array}\right)$$

We can think of A_i' as if we cut out from U the block A_i with a part of the column just after it and the suitable part of the top row. Similarly, to obtain A_i'' one cuts out from U the block A_i with the suitable part of the first column and the column just after A_i. This description is connected to the operations which we will perform on A_i' and A_i''.
The following definition is essential for the construction below.

DEFINITION 3.2. We say that the vector $\xi\in\mathbb{Z}^{n+1}$ is orthogonal to the branch of length n represented by the matrix X if $X'\xi = (1,0,\dots,0)^t$.

LEMMA 3.3. *There exists a unique vector α_i orthogonal to the i-th branch of the minimal resolution of a surface quotient singularity. It has integral and non-negative entries, which form an increasing sequence.*

PROOF. We have to determine the entries of $\alpha_i = (1, z_1, \ldots, z_{n_i})$. From the form of A_i we see that

$$\begin{aligned} z_1 &= a_{i,1} \in \mathbb{Z}, \\ z_2 &= a_{i,2} z_1 - 1 \in \mathbb{Z}, \\ z_3 &= a_{i,3} z_2 - z_1 \in \mathbb{Z}, \ldots \\ z_k &= a_{i,k} z_{k-1} - z_{k-2} \in \mathbb{Z}, \ldots \end{aligned}$$

so by induction all entries of α_i are uniquely determined and integral. Moreover, as $a_{i,j} > 1$ (they are entries of a Hirzebruch-Jung continued fraction, as defined in [**Rei97**]) it follows that the sequence (z_i) is increasing and all its elements are positive. $\square$

NOTATION 3.4. In what follows α_i will always denote the unique vector orthogonal to the i-th branch of the resolution diagram.

Now let us construct a basis of $\ker U$.

NOTATION 3.5. Any vector in $\ker U$ will be presented as quadruple (u, w_1, w_2, w_3) consisting of a number u and three vectors w_i of lengths $n_i + 1$ respectively. To write down a vector in $\ker U$ corresponding to such a quadruple, we write u as the first entry and then consequently the entries of w_1, w_2, w_3.

Such a partition is natural: when we multiply a vector of this form by U, the number u is multiplied by the numbers in the column corresponding to the branching point of the resolution diagram, and the remaining three parts correspond to the branches. Thus obviously

$$v_2 = (0, \alpha_1, 0, -\alpha_3) \quad \text{and} \quad v_3 = (0, 0, \alpha_2, -\alpha_3)$$

are in $\ker U$. We construct v_1 such that $\{v_1, v_2, v_3\}$ is a basis of $\ker U$.

LEMMA 3.6. *There is an unique vector $v \in \ker U$ of the form*

$$(1, (0, *, \ldots, *), (0, *, \ldots, *), (d, *, \ldots, *))$$

where $*$ stands for an integer and $-d$ is the self-intersection number of the central curve in the special fibre of the resolution.

PROOF. First note that the matrices A_i'' have in the kernel integral vectors of two types:

$$\beta_i = (1, 0, *, \ldots, *) \qquad \text{and} \qquad \gamma_i = (1, d, *, \ldots, *).$$

More generally, for any $a, b \in \mathbb{Z}$ there is an integral vector in the kernel of A_i'' of the form $(a, b, *, \ldots, *)$. To see this, we just use an inductive procedure as in the proof of Lemma 3.3 to determine the entries. In addition we get the conclusion that β_i and γ_i are uniquely determined, the entries of each β_i form a decreasing sequence and the entries of each γ_i form an increasing sequence.
Then look at

$$v = (1, \overline{\beta_1}, \overline{\beta_2}, \overline{\gamma_3}),$$

where $\overline{\beta_i}$ and $\overline{\gamma_i}$ are constructed from β_i and γ_i by removing the first entry, and compute $U \cdot v$. As each $\overline{\beta_i}$ starts from 0 and $\overline{\gamma_i}$ from d, we get 0 at the first entry. The remaining entries are the same as the entries of first $A_1'' \cdot \beta_1$, then $A_2'' \cdot \beta_2$ and finally $A_3'' \cdot \gamma_3$, so they are also 0.

Finally, v is uniquely determined, because if we write $v = (u, w_1, w_2, w_3)$ then $(u, (w_i)_1, \dots, (w_i)_{n_i+1})$ is in the kernel of A_i'' and is uniquely determined by u and $(w_i)_1$ for $i = 1, 2, 3$. □

NOTATION 3.7. Take $v_1 = v$ and write v_1, v_2, v_3 in the rows of a matrix K, divided into blocks as described in Notation 3.5.

$$K = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \left(\begin{array}{c|c|c|c} 1 & 0, *, \dots, * & 0, *, \dots, * & d, *, \dots, * \\ 0 & \alpha_1 & 0 & -\alpha_3 \\ 0 & 0 & \alpha_2 & -\alpha_3 \end{array}\right)$$

The choice of the matrix K defining the kernel of U is obviously non-unique; we choose one that is convenient for further computations.

REMARK 3.8. Notice that K defines the kernel of the lattice map, not only the map of vector spaces, i.e. v_1, v_2, v_3 span a full sublattice of $\mathbb{Z}^{n+3}$. As α_1 and α_2 start from 1 (by Lemma 3.3), K has an identity matrix as a minor.

3.2. The toric structure of $\mathbb{C}^{n+3}/\!\!/T$. We investigate geometric quotients of open subsets of $\mathbb{C}^{n+3}$ by the action of the Picard torus T using the toric setting, described in [**CLS11**, Chapter 5.1]. More exactly, what we do is the reverse of the construction proposed therein: instead of expressing a given toric variety as a quotient of an open set of an affine space, we reconstruct this variety and the open set knowing the torus action on an affine space.
Thus we think of T as of a subtorus of the big torus $(\mathbb{C}^*)^{n+3} \subset \mathbb{C}^{n+3}$. First look at the map of tori:

$$0 \to T \xrightarrow{\varphi|_{T\times\{(1,\dots,1)\}}} (\mathbb{C}^*)^{n+3} \to (\mathbb{C}^*)^3 \to 0,$$

Let $M' \simeq \mathbb{Z}^{n+3}$ and $M \simeq \mathbb{Z}^3$ be the lattices of characters of the big torus in $\mathbb{C}^{n+3}$ (with the same fixed coordinates) and of the quotient torus respectively. By P we denote the monomial lattice of T (which in fact can be identified with the Picard group of X). Then we have a map of monomial lattices

$$0 \to M \to M' \xrightarrow{U} P \to 0$$

and we can identify M with $\ker U \subset M'$. By K^t we will denote the matrix (or the corresponding lattice map) of the embedding $M \to M'$.
Thus we have described the big torus of the quotient variety. To understand the whole structure we first determine the set of rays of its fan and then look which points have to be removed from $\mathbb{C}^{n+3}$ to obtain in a quotient a variety corresponding to a certain simplicial fan with this set of rays. (In fact these are unstable points with respect to some linearization of the action.) Dualize the map of monomial lattices:

$$0 \to P^\vee \xrightarrow{U^t} N' \xrightarrow{K} N \to 0$$

(it is exact on both ends as M is a sublattice of M').
We are interested in the third arrow. The fan Σ' of $\mathbb{C}^{n+3}$ in N' is just the positive orthant and all its faces. Our aim is to describe the fans of the quotient varieties – such a fan, when we choose one of the possible geometric quotients, will be denoted by Σ. The rays of Σ are the images of the rays of Σ' under the map given by K. Just from these facts we can derive some more information on the structure of these fans, but some part of this structure varies between different models of the quotient.

First of all, we look at the rays of the quotient fans. Their coordinates are given in the columns of K. Let x, y, z be the coordinates in $N_{\mathbb{R}} = N \otimes \mathbb{R}$ corresponding to the standard basis in N.

Remark 3.9. Let us look for a moment at the analogical set-up for a quotient $\mathbb{C}^2/H$ by an abelian group $H < GL(2, \mathbb{C})$. More exactly, let a_j, $j = 1, \ldots, k$ be the entries of a Hirzebruch-Jung continued fraction whose entries are the self-intersection numbers of components of the special fibre of the minimal resolution X of $\mathbb{C}^2/H$. Let A be a matrix constructed from the sequence (a_j) in the same way as the matrix A_i corresponding to a branch of the diagram: with $-a_j$ on the diagonal, 1 just below and just above the diagonal and 0 in other entries. Then the matrix defining the homomorphism $\ker A'' \hookrightarrow \mathbb{Z}^{k+2}$, constructed by taking two (general enough) vectors in the kernel of A'' as its rows, corresponds to the toric quotient of $\mathbb{C}^{n+2}$ by the Picard torus action. In other words, the columns of this matrix are rays of the fan of the minimal resolution of $\mathbb{C}^2/H$ (in some chosen coordinates). In particular, a pair of adjacent columns is a lattice basis, which will be used later on. (The details can be established based on [**CLS11**, Section 5.2] together with [**Rei97**].)

Lemma 3.10. *The images of the rays of Σ' in N are divided into three groups, corresponding to the branches of the diagram, of vectors lying in three planes: $y = 0$, $z = 0$ and $y = z$. The intersection of these planes is the line $y = z = 0$, represented by the central ray $(1, 0, 0)$, the first column of K. Moreover, the rays in each group together with $(1, 0, 0)$, considered in the plane containing them, form the 1-skeleton of a fan of the minimal resolution of a cyclic quotient singularity.*

Proof. Only the last statement is non-obvious. As in the proof of Lemma 3.6, $v_1 = (1, \overline{\beta_1}, \overline{\beta_2}, \overline{\gamma_3})$, where $\overline{\beta_i}$, $\overline{\gamma_i}$ are β_i, γ_i without the first entry, and β_i, γ_i are vectors in the kernel of A_i'' starting from $(1, 0)$ and $(1, d)$ respectively. Note that also vectors α_i' obtained from α_i by adding 0 at the beginning are in the kernel of A_i''. Hence the kernel of A_i'' is spanned by α_i' and β_i or α_i' and γ_i (A_i is nonsingular, because if the numbers on its diagonal come from the Hirzebruch-Jung continued fraction of p/q then $\det A_i = \pm p$). Remark 3.9 implies that the dual of $\ker A_i'' \hookrightarrow \mathbb{Z}^{n_i+2}$ map the fan of the affine space to the fan of the minimal resolution of the respective cyclic quotient singularity, so if we write any pair of generators of its kernel as the rows of a matrix, the columns will be the rays of the resolution fan. And there are isomorphisms (of lattices and fans) between the results of this construction for the pairs (β_1, α_1'), (β_2, α_2'), $(\gamma_3, -\alpha_3')$ and the fans given the images of rays of Σ' in one of the planes $y = 0$, $z = 0$, $y = z$ and the cones spanned by the pairs of adjacent rays. $\square$

Definition 3.11. By the *outer rays* of Σ we understand the set consisting of three rays, the last one from each group (i.e. the furthest from the central ray $(1, 0, 0)$).

Lemma 3.12. *Assume that the numbers at the vertices of the i-th branch of the minimal resolution diagram are the negatives of the entries of a Hirzebruch-Jung continued fraction p_i/q_i and $-d$ is associated to the branching point. Then the outer rays are*

$$(dp_3 - q_3, -p_3, -p_3), \quad (-q_2, 0, p_2), \quad (-q_1, p_1, 0).$$

Proof. We have to find some formulae for the last entries of vectors $\alpha_i, \beta_i, \gamma_i$ (introduced in Lemmata 3.3 and 3.6). By $[[z_1, z_2, \dots, z_k]]$ we denote the Hirzebruch-Jung continued fraction with the entries $z_1, z_2, \dots, z_k$.
First of all we notice that the recursive formula for the entries of α_i (in the proof of Lemma 3.3) provides in fact a procedure of computing the numerator of the reversed continued fraction, i.e. if $p_i/q_i = [[a_{i,1}, \dots, a_{i,n_i}]]$, then using this formula one obtains the numerator of $[[a_{i,n_i}, \dots, a_{i,1}]]$. But the reversed continued fraction to p_i/q_i is p_i/q_i' where q_i' is reverse modulo p to q_i (this and other useful facts on Hirzebruch-Jung continued fractions can be found in [**CLS11**, Section 10.2]). Thus α_i ends with p_i.
If we write down an analogous formula for β_i, we obtain that its last entry is the negative of the numerator of $[[a_{i,n_i}, \dots, a_{i,2}]]$, which is the same as the negative of the numerator of $[[a_{i,2}, \dots, a_{i,n_i}]] = r_i/s_i$. But

$$\frac{p_i}{q_i} = a_{i,1} - \frac{1}{\frac{r_i}{s_i}},$$

so indeed $r_i = q_i$ and β_i ends with $-q_i$.
Let α_i' be α_i with 0 added as a first entry. Then $\gamma_i = \beta_i + d\alpha_i'$, because each of them is uniquely determined by the first two entries (as shown in Lemma 3.6) and these entries satisfy this equality. Therefore γ_i ends with $dp_i - q_i$. □

Remark 3.13. Let $G = BD_{n,m}$. Then using [**Bri68**, Satz 2.11] one can see that the last entry of γ_i can be written as $m + p_3$, so the first outer ray is $(m+p_3, -p_3, -p_3)$.

Lemma 3.14. *The outer rays span a convex cone which contains* $(1,0,0)$ *inside.*

Proof. We show that $(1,0,0)$ is a positive combination of the outer rays. We have

$$\frac{p_3}{p_1}(-q_1, p_1, 0) + \frac{p_3}{p_2}(-q_2, 0, p_2) + (dp_3 - q_3, -p_3, -p_3) = p_3\left(d - \frac{q_3}{p_3} - \frac{q_1}{p_1} - \frac{q_2}{p_2}\right)(1,0,0),$$

so it suffices to prove that

$$d - \frac{q_1}{p_1} - \frac{q_2}{p_2} - \frac{q_3}{p_3} > 0.$$

If $d \geq 3$ this is obvious, because $q_i < p_i$. And if $d = 2$, this can be checked case by case using the table in [**Bri68**, Satz 2.11]. The cases where this number is smallest are the quotient by the subgroups of $SL(2, \mathbb{C})$. □

Notation 3.15. We are interested in fans $\Sigma \subset N_{\mathbb{R}}$ with the set of rays as described above and such that the sum of all cones in Σ is the convex cone spanned by the outer rays. We consider only simplicial fans. From now on Σ will denote a fan satisfying these conditions.

The choice of such a fan corresponds to the choice of the quotient X_Σ of an open subset of $\mathbb{C}^{n+3}$ by T. More precisely, X_Σ is a geometric quotient of $\mathbb{C}^{n+3} \setminus Z(\Sigma)$ by T, where $Z(\Sigma)$ is the zero set of the irrelevant ideal of Σ (see for example [**CLS11**, Theorem 5.1.11]; the quotients are geometric because we consider only simplicial fans). We study the structure of $Z(\Sigma)$ in section 4.1.
It turns out that some 2-dimensional faces have to be in Σ, independently of the choice.

Lemma 3.16. Σ *contains the following two-dimensional faces:*

(1) *all faces spanned by two adjacent rays in one of the planes* $y = 0$*,* $z = 0$*,* $y = z$*,*
(2) $\sigma((0,1,0),(0,0,1))$*,* $\sigma((0,1,0),(d,-1,-1))$*,* $\sigma((0,0,1),(d,-1,-1))$*.*

Moreover, the cones containing the central ray are smooth and the divisor associated to the central ray is a $\mathbb{P}^2$*.*

Proof. First assume that the cone spanned by two adjacent rays ρ and ρ' from one branch is not in Σ. Let ρ be nearer to the central ray $(1,0,0)$. Then there exists a cone in Σ spanned by ρ, ρ_1, ρ_2 such that each of these rays comes from a different branch. But such a cone contains $(1,0,0)$ in the interior, because $(1,0,0)$ is inside the cone spanned by the outer rays and all the rays lie on three planes containing it, which is impossible.
As for the cones in (2), they must be in Σ, as the rays $(0,1,0)$, $(0,0,1)$ and $(d,-1,-1)$ are the only three which can span a face with $(1,0,0)$. The central ray must be in the interior of the cone spanned by the outer rays, so it belongs to at least three cones of maximal dimension.
If we choose any pair of vectors from $(0,1,0)$, $(0,0,1)$, $(d,-1,-1)$ and take $(1,0,0)$ as the third one, we have a triple that generates the whole lattice, so the central part of our picture consists of three smooth cones.
To describe the structure of the divisor associated to the central ray $(1,0,0)$ of this fan one has to project these rays to the orthogonal plane $x = 0$ (see [**CLS11**, Proposition 3.2.7]). The result is the fan with $(1,0)$, $(0,1)$ and $(-1,-1)$ as rays and all cones generated by a proper subset of the set of rays, hence the fan of a smooth $\mathbb{P}^2$.

□

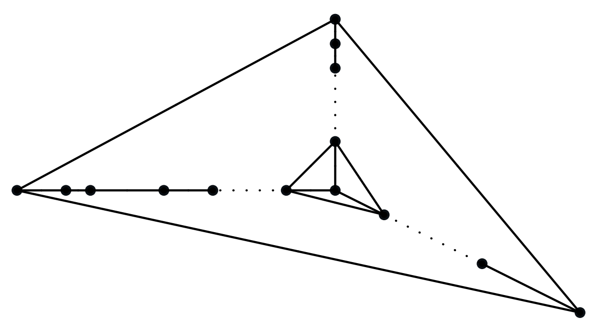


Figure 2. Faces that have to be in Σ (shown in a section)

To summarize, the fans we consider are simplicial fans such that their cones sum up to the cone spanned by the outer rays. Fig. 2 is a schematic picture of a section of the cone spanned by the outer rays with the sections of faces mentioned in Lemma 3.16 included. All considered fans correspond to triangulations of such a diagram. Toric varieties obtained this way are different geometric quotients of open subsets of $\mathbb{C}^{n+3}$ by T. In general there is no smooth model. For example, the cones adjacent to the faces of the big cone spanned by the outer rays are most often non-smooth.

3.3. The candidate for $\mathrm{Spec}(\mathrm{Cox}(X))$**.** We introduce a hypersurface $S \subset \mathbb{C}^{n+3}$, which is our candidate for the spectrum of the Cox ring of the minimal

resolution of $\mathbb{C}^2/G$. Its equation can be determined from the resolution diagram together with the self-intersection numbers of the components of the special fibre.

CONSTRUCTION 3.17. We define a hypersurface $S \subset \mathbb{C}^{n+3}$ by describing its ideal

$$I(S) \subset \mathbb{C}[y_0, y_{1,1}, \ldots, y_{1,n_1}, x_1, y_{2,1}, \ldots, y_{2,n_2}, x_2, y_{3,1}, \ldots, y_{3,n_3}, x_3],$$

which is generated by a trinomial equation, each monomial of which corresponds to one branch of the resolution diagram. The variables, except y_0, are divided into three sequences

$$(y_{i,1}, y_{i,2}, \ldots, y_{i,n_i-1}, y_{i,n_i}, x_i)$$

for $i = 1, 2, 3$, and all variables in the i-th sequence appears only in the monomial corresponding to the i-th branch. The equation of S is the sum of these three monomials.

As the i-th vector of exponents we take the vector α_i orthogonal to the i-th branch, so the equation is

$$\sum_{i=1,2,3} y_{i,1}^{(\alpha_i)_1} \cdots y_{i,n_i}^{(\alpha_i)_{n_i}} \cdot x_i^{(\alpha_i)_{n_i+1}} = 0.$$

REMARK 3.18. As all entries of each α_i are positive integers and the first one is 1 (see Lemma 3.3), this polynomial is well-defined and the variables $y_{1,1}, y_{2,1}, y_{3,1}$ appear in the equation with exponent 1.

The choice of coefficients of monomials equal to 1 is arbitrary. For any other set of coefficients we just obtain a different embedding of $\mathrm{Spec}(\mathrm{Cox}(X))$ in $\mathbb{C}^{n+3}$.

EXAMPLE 3.19. In the case of Du Val singularities the equation is formed as follows: for each variable its exponent is equal to the distance of the corresponding vertex in the resolution diagram from the branching point (as x_i corresponds to the added leaf of the i-th branch, its distance from the branching point is the distance of y_{i,n_i} plus 1). For example let us look at E_8 singularity. The extended intersection matrix is

$$U(BI) = \left(\begin{array}{c|cc|ccc|ccccc}
-2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1
\end{array}\right)$$

and the kernel matrix with the rays of $\Sigma(BI)$ as columns is

$$K(BI) = \left(\begin{array}{c|cc|ccc|ccccc}
1 & 0 & -1 & 0 & -1 & -2 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & 2 & 0 & 0 & 0 & -1 & -2 & -3 & -4 & -5 \\
0 & 0 & 0 & 1 & 2 & 3 & -1 & -2 & -3 & -4 & -5
\end{array}\right)$$

The entries of vectors α_i, which are the exponents in the equation, can be read out from the second and third row of $K(BI)$:

$$S(BI) = \{y_{1,1}x_1^2 + y_{2,1}y_{2,2}^2x_2^3 + y_{3,1}y_{3,2}^2y_{3,3}^3y_{3,4}^4x_3^5 = 0\}$$

Example 3.20. Let us look at a group which is not in $SL(2,\mathbb{C})$: take $BD_{23,39}$, which appeared already in Example 2.6. We have

$$U(BD_{23,39}) = \left(\begin{array}{c|cc|cc|ccccc} -3 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 1 \end{array}\right)$$

$$K(BD_{23,39}) = \left(\begin{array}{c|cc|cc|ccccc} 1 & 0 & -1 & 0 & -1 & 3 & 11 & 19 & 27 & 62 \\ 0 & 1 & 2 & 0 & 0 & -1 & -4 & -7 & -10 & -23 \\ 0 & 0 & 0 & 1 & 2 & -1 & -4 & -7 & -10 & -23 \end{array}\right)$$

and again we read out vectors α_i from $K(BD_{23,39})$ obtaining

$$S(BD_{23,39}) = \{y_{1,1}x_1^2 + y_{2,1}x_2^2 + y_{3,1}y_{3,2}^4 y_{3,3}^7 y_{3,4}^{10} x_3^{23} = 0\}.$$

Lemma 3.21. *The hypersurface S is invariant under the action of T defined by formula 3.0.1.*

Proof. The weights of the action are given by the columns of U, so to compute the weight vector of the action on the monomial corresponding to the i-th branch one multiplies U by $(0,\alpha_1,0,0)$, $(0,0,\alpha_2,0)$ or $(0,0,0,\alpha_3)$ respectively. Because α_i is orthogonal to the i-th branch we obtain that T acts on each monomial, and therefore on the whole equation, by multiplication by t_0. Thus the set of zeroes of this equation is invariant under the action of T. $\square$

Therefore we may consider geometric quotients of open subsets of S by T. They will be presented as subsets in different geometric quotients of open sets in $\mathbb{C}^{n+3}$ by T.

4. The resolution as a divisor in a toric variety

The aim of this section is to describe important properties of certain geometric quotients of open subsets of $S \subset \mathbb{C}^{n+3}$, introduced in Construction 3.17, by the Picard torus action investigated above. Fix a simplicial fan $\Sigma \subset \mathbb{R}^3$ with the set of rays given in columns of the matrix K, as described in section 3.2. We consider an open subset of S obtained by removing zeroes of the irrelevant ideal

$$W = S \setminus Z(\Sigma) \subset \mathbb{C}^{n+3} \setminus Z(\Sigma)$$

and its quotient by the action of T.

Remark 4.1. Since the quotient X_Σ of $\mathbb{C}^{n+3} \setminus Z(\Sigma)$ by T is geometric and $W = S \setminus Z(\Sigma)$ is a T-invariant closed subset of $\mathbb{C}^{n+3} \setminus Z(\Sigma)$, the quotient of W by T is also geometric (e.g. by [**ADHL10**, Proposition 2.3.9]).

We investigate the quotient $Y = W/T$ by looking at the embeddings which are horizontal arrows in the following diagram and using toric geometry.

$$\begin{tikzcd}
W = S \setminus Z(\Sigma) \arrow[r, hook] \arrow[d, "/T"] & \mathbb{C}^{n+3} \setminus Z(\Sigma) \arrow[d, "/T"] \\
Y = W/T \arrow[r, hook] & X_\Sigma
\end{tikzcd}$$

The first thing we prove is the smoothness of Y (see Proposition 4.5), which follows from the fact that the action of T on W is free. In section 4.2 we construct a birational morphism from Y to the quotient $\mathbb{C}^2/G$, coming from the embedding in a toric variety, which implies that Y is a resolution of the singularity from which we have started. However, at this point we do not know if it is the minimal resolution. This is proven in section 4.3 by computing intersection numbers of the irreducible components of the exceptional divisor of the constructed morphism to $\mathbb{C}^2/G$.

4.1. Smoothness of the quotient. The aim of this section is to prove Proposition 4.5, which states that the quotient $Y = W/T$ is smooth. In order to give the proof we first need to analyze the structure of the set $Z(\Sigma)$ of zeroes of the irrelevant ideal associated to the chosen fan Σ.

LEMMA 4.2. *The set $W = S \setminus Z(\Sigma) \subset \mathbb{C}^{n+3}$ consists of three types of points:*

(1) *points with all coordinates nonzero,*
(2) *points with one coordinate equal to zero,*
(3) *points with two zero coordinates: either corresponding to a pair of adjacent rays on one branch of the diagram, or corresponding to the central ray and the first ray on one branch.*

Hence, in fact, W is independent of the choice of Σ.

PROOF. The argument is a straightforward analysis of the structure of the irrelevant ideal $B(\Sigma) = \langle x^{\hat{\sigma}} \, : \, \sigma \in \Sigma_{max}\rangle$.
Recall that $x^{\hat{\sigma}}$ is the product of variables corresponding to all the rays in $\Sigma(1)$ that are not in $\sigma(1)$ and Σ_{max} in our case consists of 3-dimensional cones of Σ. Let us look at the number of coordinates equal to zero in a point in $Z(\Sigma)$.
First of all, if a point has ≥ 4 zeroes on different coordinates, or 2 or 3 zeroes on coordinates corresponding to the rays whose images do not span a cone in Σ, then it obviously belongs to $Z(\Sigma)$.
If a point $p \in S$ has 3 zeroes on coordinates corresponding to the rays whose images span a cone in Σ, then these rays come from two different branches. But then in the equation of S the monomial corresponding to the third branch also is 0, so there must be at least one more coordinate equal to zero. Thus $p \in Z(\Sigma)$, which implies that W does not contain any point with ≥ 3 zeroes.
The same argument works in the case where p has 2 zeroes on the coordinates corresponding to the rays from two different branches. Thus p with 2 zeroes can belong to W only if these zeroes are on the coordinates corresponding to adjacent rays from one branch or to the central ray and the first ray on a branch. □

We need a following technical observation to prove that the action of T on W is free in Lemma 4.4.

LEMMA 4.3. *If we remove from the extended intersection matrix U any two columns corresponding to a pair of adjacent vertices on one branch of the resolution diagram, the remaining ones generate $\mathbb{Z}^n$.*

Proof. We say that a column of U is in the j-th set if it intersects the block A_j or lies just after it (hence the first column does not belong to any of these sets). Recall that A_i is a square matrix with n_i rows. By $e_1, \ldots, e_n$ we denote the standard basis of $\mathbb{Z}^n$.
The first case is when we remove two adjacent columns from one set; without loss of generality assume that they lie in the third set. Performing reductions with integral coefficient on columns from the first set, in the same way as in the proof of Lemma 3.3, but starting from the last column, we obtain $e_1, e_2, \ldots, e_{n_1+1}$. From the second set we get e_1 and $e_{n_1+2}, \ldots, e_{n_1+n_2+1}$. Then, using the first column, we have also $e_{n_1+n_2+2}$, and e_n appears at the end of the third set, if it is not removed. Now we can perform the reductions on the columns from the third set starting both from the first and the last column (if the first two or the last two columns are removed, we start from one end only). It can be easily seen that this way we obtain all the remaining vectors of the standard basis of $\mathbb{Z}^n$.
The second case, which is removing the first column of U and the first column in one set, is even easier. By linear combinations with integral coefficients, from the two complete sets we obtain all corresponding standard basis vectors, including e_1, which is later used to construct the remaining basis vectors from the set with the removed column. $\square$

Lemma 4.4. *T acts freely on W.*

Proof. We have to check that a point $p \in W$ cannot have nontrivial isotropy group. Assume that $\underline{t} = (t_0, \ldots, t_{n-1}) \in T$ is such that $\underline{t}p = p$. This means that all characters defining the action, except these corresponding to the coordinates equal to zero in p, give 1 evaluated at $\underline{t}$.
Our aim is to show that $t_i = 1$ for $i = 0, \ldots, n-1$. Because p is of one of three types listed in Lemma 4.2, this can be reformulated as follows: if we remove from U the columns corresponding to the zero coordinates in p then the remaining columns span the lattice $\mathbb{Z}^n \subset \mathbb{C}^n$. And this result follows directly from Lemma 4.3. $\square$

Proposition 4.5. *The quotient $Y = W/T$ is smooth.*

Proof. We prove that W is smooth by checking that all the singular points of S are in $Z(\Sigma)$. Indeed, if the Jacobian of the equation of S is zero in a point $(y_0, y_{1,1}, \ldots, x_1, y_{2,1}, \ldots, x_2, y_{3,1}, \ldots, x_3)$ then for each branch of the diagram at least one of the coordinates corresponding to its vertices is zero. Hence there are at least three coordinates equal to zero and, by Lemma 4.2, such a point is not in W.
Since Y is a geometric quotient of a smooth variety by a free action of T, it is also smooth. A standard reference for such a statement is Luna's slice theorem, but we believe that this particular case can be much simpler. By the classical result of Sumihiro [**Sum74**] any point $w \in W$ has a T-invariant affine neighborhood and by applying Luna's theorem [**Lun73**] to this neighborhood we know that the quotient is smooth in the image of w. $\square$

4.2. The quotient is a resolution of $\mathbb{C}^2/G$. An embedding of investigated geometric quotient $Y = W/T$ in a toric variety X_Σ leads to a construction of a birational morphism $Y \to \mathbb{C}^2/G$, shown below.

Proposition 4.6. *There exists a birational morphism from W/T onto $\mathbb{C}^2/G$.*

PROOF. Let Δ denote the fan (considered in lattice N) consisting of a cone spanned by the outer rays of Σ and all its faces. Look at the composition π of two fan morphisms: $\Sigma' \to \Sigma$, given by the matrix K (as in Notation 3.7) and $\Sigma \to \Delta$, induced by the identity on N. We would like to describe the image of S in X_Δ under π.
As Δ is simplicial, X_Δ is a quotient of $\mathbb{C}^3$ by a finite group action. Let $N'' \simeq \mathbb{Z}^3$ and Γ be the fan consisting of the positive octant in N'' and all its faces. Then $\omega : N'' \to N$ which sends the standard basis to the rays of Δ is the toric description of this quotient map. But the embedding $\eta : N'' \hookrightarrow N'$ which maps the standard basis to the rays corresponding to the variables x_1, x_2, x_3 commutes with π and ω. In coordinates corresponding to the standard bases η is just the embedding of $\mathbb{C}^3$ by x_1, x_2, x_3 to the subspace defined by $y_0 = 1$ and $y_{i,j} = 1$ for all possible i, j. Therefore the restriction of S to $X_\Gamma \simeq \mathbb{C}^3$ with coordinates x_1, x_2, x_3 is given by the equation obtained from the equation of S by leaving x_1, x_2, x_2 without change and substituting 1 for all other variables:

$$x_1^{p_1} + x_2^{p_2} + x_3^{p_3} = 0.$$

Looking at the table [**Bri68**, p. 348] we obtain the following equations of the restriction of S to X_Γ:

$$\begin{array}{ll} BD_{n,m} & x_1^2 + x_2^2 + x_3^n = 0 \\ BT_m & x_1^2 + x_2^3 + x_3^3 = 0 \\ BO_m & x_1^2 + x_2^3 + x_3^4 = 0 \\ BI_m & x_1^2 + x_2^3 + x_3^5 = 0 \end{array}$$

Comparing with Lemma 2.2 and [**Rei**, Table 1] we see that for a group G the equation above is just an equation of the quotient by the commutator subgroup of G. (For BD_2, i.e. the commutator subgroup of BT_m, the equation is most often given in the form $x_1^2 + x_2^3 + x_2 x_3^2 = 0$, but it is the same up to a change of coordinates.)
X_Δ is a quotient of $\mathbb{C}^3$ by an action of a finite group $J = \operatorname{coker} \omega$. The image of S in X_Δ is then the quotient of the restriction of S to X_Γ by J. We can write ω in the standard basis using the matrix with the outer vectors of Σ as columns. In all cases it is easy to check that J is isomorphic to the abelianization of G – one has to use the parameters of the resolution given in [**Bri68**, p. 348] and Corollary 2.3. Our aim is to prove that the the quotient of $S \cap X_\Gamma$ by J is isomorphic to $\mathbb{C}^2/G \simeq (\mathbb{C}^2/[G,G])/Ab(G)$. Thus we have to argue that the action of J on $S \cap X_\Gamma$ is the same as the action of $Ab(G)$ on $\mathbb{C}^2/[G,G]$. We do this by comparing the action on the coordinate rings: the weights of the action of generators of J on the chosen coordinates of X_Γ have to be identical to the weights of the action of the corresponding generators of $Ab(G)$ on the $[G,G]$-invariants which satisfy the equation of $S \cap X_\Gamma$.
The action of $Ab(G)$ on the invariants of $[G,G]$ is easy to describe. The generators of these rings of invariants are listed for example in [**DZ93**]. We need a set of generators which are eigenvectors of the action of $Ab(G)$, because the coordinates of X_Γ satisfy this condition. In most of the cases the invariants given in [**DZ93**] are eigenvectors of $Ab(G)$ (and there is no other choice), only in the case of BT_m, where the commutator subgroup is BD_2, one has to take suitable linear combinations of $x^4 + y^4$ and $x^2 y^2$. (The $[G,G]$-invariants which are eigenvectors of the

action of $Ab(G)$ are considered also in section 6; in fact we list them in Example 6.11.) Finally, we take some representatives of the generating classes of $Ab(G)$ and determine their action on the chosen invariants by an explicit computation.
To compute the weight of the action of J on the variables x_1, x_2, x_3 corresponding to the rays of Γ we take a vector representing a generator and evaluate it on the dual characters to the rays of Δ, which are

$$\chi_1 = \frac{1}{r}(p_1p_2, q_1p_2, q_2p_1),$$
$$\chi_2 = \frac{1}{r}(p_1p_3, q_1p_3, dp_1p_3 - p_1q_3 - q_1p_3),$$
$$\chi_3 = \frac{1}{r}(p_2p_3, dp_2p_3 - p_2q_3 - q_2p_3, q_2p_3),$$

where r is the order of J (so the determinant of the matrix which has the outer rays as columns)

$$r = dp_1p_2p_3 - q_1p_2p_3 - p_1q_2p_3 - p_1p_2q_3.$$

In all the cases of cyclic abelianizations, as a generator of J one can take one of the standard basis vectors. In the only non-cyclic case the generators can be for example $(0, 1, 0)$ and $(-1, 1, 1)$. However, these generators do not necessarily give the same weights as the action of $Ab(G)$ on the chosen $[G, G]$-invariants, so one has to find a suitable power of a generator to get the same numbers. Such generators exist in all the cases; all the computations are very similar, so we present only one case in detail.
Let us look at the action of $G = BO_m$; note that we have to assume $(m, 6) = 1$. First, the generators of the invariants of $[G, G] = BT$ are

$$w_3 = x^5y - xy^5,$$
$$w_2 = x^8 + 14x^4y^4 + y^8,$$
$$w_1 = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12},$$

which, up to some constants, satisfy the relation $w_1^4 + w_2^3 + w_3^2 = 0$. As stated in Corollary 2.3, $Ab(G) \simeq \mathbb{Z}_{2m}$ is generated by $g = \varepsilon_{2m} \cdot diag(\varepsilon_8, \varepsilon_8^{-1})$. The action on the $[G, G]$-invariants is

$$g \cdot w_1 = -\varepsilon_{2m}^6 \cdot w_1 = \varepsilon_{2m}^{m+6} \cdot w_1,$$
$$g \cdot w_2 = \varepsilon_{2m}^8 \cdot w_2,$$
$$g \cdot w_3 = -\varepsilon_{2m}^{12} \cdot w_3 = \varepsilon_{2m}^{m+12} \cdot w_3.$$

Now look at the action of J. Take $v = (0, 1, 0) \in N$. Then

$$\chi_1(v) = \frac{3}{r} = \frac{3}{2m}$$

so, as m is not divisible by 3, v is of order $2m$ in J, so it is a generator. The weights of its action are

$$2m \cdot \chi_1(v) = 3,$$
$$2m \cdot \chi_2(v) = 4,$$
$$2m \cdot \chi_3(v) = 12d - 3q_3 - 4q_2 = m + 6.$$

If we take $v' = (0, m+2, 0) \in N$, which is also a generator of J, as m is odd, then

$$\begin{aligned}
2m \cdot \chi_1(v') &= 3(m+2) \equiv m+6 \mod 2m,\\
2m \cdot \chi_2(v') &= 4(m+2) \equiv 8 \mod 2m,\\
2m \cdot \chi_3(v') &= (m+6)(m+2) = m^2 + 12 + 8m \equiv m+12 \mod 2m,
\end{aligned}$$

hence both considered actions are the same.

□

COROLLARY 4.7. *The quotient $(S \backslash Z(\Sigma)) /\!/ T$ is a resolution of the singularity $\mathbb{C}^2/G$.*

4.3. Minimality. We prove that Y is in fact the minimal resolution of considered quotient singularity. Moreover, we explain how the class groups of Y and X_Σ are related, which will be needed in the proof of Proposition 5.2.

PROPOSITION 4.8. *Y is the minimal resolution of $\mathbb{C}^2/G$. Moreover, $\mathrm{Cl}(Y)$ is generated by the restrictions to Y of divisors in X_Σ which are invariant under the action of the big torus. The intersection numbers of these divisors and the exceptional curves are the entries of the extended intersection matrix U.*

PROOF. Let $D_{i,j}$ be the divisor in X_Σ corresponding to $\rho_{i,j}$ (the j-th ray on the i-th branch) and D_0 is the divisor corresponding to the central ray ρ_0. Note that $C_0 = Y \cap D_0$ and $C_{i,j} = Y \cap D_{i,j}$ for $j < n_i$ are the exceptional curves of the map from Y to the toric variety corresponding to the cone spanned by the outer rays of Σ.

By Lemma 4.2 W does not depend on the choice of Σ, so we can investigate each exceptional curve $C_{i,j}$ in a fan where each of $\rho_{i,j-1}$, $\rho_{i,j}$ and $\rho_{i,j+1}$ lies in 2-dimensional faces with the first rays on two other branches. It is straightforward to check that in this situation $D_{i,j}$ is isomorphic to the Hirzebruch surface $F_{(\gamma_i)_j}$, where $(\gamma_i)_j$ is the j-th entry of the vector $\gamma_i \in \ker A''_i$ starting from 1 and d (see the proof of Lemma 3.6). In order to compute the intersection number of C_{i,n_i}, which is not exceptional, with C_{i,n_i-1} we consider the fan where ρ_{i,n_i-1} and ρ_{i,n_i} form (smooth) cones with the first rays on two other branches.

One can compute local equation of Y on affine pieces of X_Σ by localizing equations in the Cox ring of a toric variety to open sets corresponding to cones of maximal dimension, as shown in [**CLS11**, Prop. 5.2.10]. In these local coordinates it is easy to check that $C_{i,j} \cdot C_{i,j+1} = 1$ and $C_0 \cdot C_{i,1} = 1$ in Y for all admissible i, j. We now compute $C_{i,j} \cdot C_{i,j}$ in Y.

Look at $D_{i,j} \simeq F_{n_{i,j}}$ as at a $\mathbb{P}^1$-fibration over $\mathbb{P}^1$, the structure of which is given by Σ. Passing to local coordinates again, we check that $C_{i,j}$ is a fibre of this fibration. However, it is not one of the fibres which are torus invariant curves in X_Σ and correspond to the 2-dimensional faces joining $\rho_{i,j}$ with the first rays on the other branches. It can also be seen in the local coordinates that Y intersects $D_{i,j}$ transversally in $C_{i,j}$.

Let ι and κ denote the embedding of Y and $D_{i,j}$ in X_Σ respectively. By the projection formula

$$\iota_*(\iota^* \mathcal{O}_{X_\Sigma}(D_{i,j}) \cap C_{i,j}) = \mathcal{O}_{X_\Sigma}(D_{i,j}) \cap \iota_* C_{i,j}.$$

As $\iota^* \mathcal{O}_{X_\Sigma}(D_{i,j})$ is just $D_{i,j} \cap Y = C_{i,j}$, the left hand side is just the self-intersection number of $C_{i,j}$ in Y. And the right hand side is $D_{i,j} \cdot C_{i,j}$ in X_Σ.

Let $C'_{i,j}$ be one of the fibres in $D_{i,j} \simeq F_{n_{i,j}}$ which is a torus invariant curve in X_Σ. Assume that it corresponds to the face $\sigma(\rho_{i,j}, \rho_{k,1})$. Then

$$\begin{aligned}\mathcal{O}_{X_\Sigma}(D_{i,j}) \cap \iota_* C_{i,j} = \mathcal{O}_{X_\Sigma}(D_{i,j}) \cap \kappa_* C_{i,j} = \kappa_*(\kappa^* \mathcal{O}_{X_\Sigma}(D_{i,j}) \cap C_{i,j}) = \\ = \kappa_*(\kappa^* \mathcal{O}_{X_\Sigma}(D_{i,j}) \cap C'_{i,j}) = \mathcal{O}_{X_\Sigma}(D_{i,j}) \cap \kappa_* C'_{i,j}.\end{aligned}$$

Hence instead of $D_{i,j} \cdot C_{i,j}$ we compute $D_{i,j} \cdot C'_{i,j}$ in X_Σ.
From [**CLS11**, Prop. 6.3.8] we know that if m_1 and m_2 are Cartier data of $D_{i,j}$ for $\sigma(\rho_{k,1}, \rho_{i,j-1}, \rho_{i,j})$ and $\sigma(\rho_{k,1}, \rho_{i,j}, \rho_{i,j+1})$ respectively then

$$D_{i,j} \cdot C'_{i,j} = \langle m_1 - m_2, \rho_{i,j+1} \rangle.$$

We show the computations in the case where $i = 3$ and $k = 1$, other cases can be reduced to this one by applying a lattice automorphism. For $1 \le p \le n_3$ we have

$$\rho_{1,1} = (0, a, 0), \quad \rho_{3,p} = (b_p, c_p, c_p).$$

Moreover, as adjacent rays on each branch form a basis of the lattice restricted to the subspace spanned by them (see Lemma 3.10), we know that $b_p c_{p-1} - b_{p-1} c_p = 1$. Hence

$$m_1 = (c_{j-1}, 0, -b_{j-1}), \quad m_2 = (-c_{j+1}, 0, b_{j+1})$$

and thus

$$D_{3,j} \cdot C'_{3,j} = c_{j-1} b_{j+1} - b_{j-1} c_{j+1}.$$

As b_p, c_p are entries of vectors $\overline{\gamma_3}$ and $-\alpha_3$ in the kernel of A'' (see Lemma 3.6 and the definition of the matrix K just after it), they satisfy recursive relations $b_{p+1} = -(a_{3,p} b_p + b_{p-1})$ and $c_{p+1} = -(a_{3,p} c_p + c_{p-1})$ with $b_0 = 1$, $b_1 = d$, $c_0 = 0$, $c_1 = 1$. Hence we check that

$$\begin{aligned}c_{j-1} b_{j+1} - b_{j-1} c_{j+1} = -c_{j-1}(a_{3,j} b_j + b_{j-1}) + b_{j-1}(a_{3,j} c_j + c_{j-1}) = \\ = -a_{3,j}(c_{j-1} b_j - b_{j-1} c_j) = -a_{3,j},\end{aligned}$$

and in general

$$C_{i,j} \cdot_S C_{i,j} = D_{3,j} \cdot_{X_\Sigma} C'_{3,j} = -a_{i,j}.$$

In a very similar way we compute $C_0 \cdot C_0$, which is equal to the intersection number of D_0 with the curve corresponding to one of the cones $\sigma(\rho_0, \rho_{i,1})$. Thus we obtain $C_0 \cdot C_0 = -d$.
Therefore Y is the minimal resolution of $\mathbb{C}^2/G$ and the intersection numbers of the exceptional curves C_0, $C_{i,j}$ for $j < n_i$ with the divisors C_0, $C_{i,j}$ for $j \le n_i$ are just the entries of the matrix U, defined in Notation 2.7.
To prove that the restrictions of the torus invariant divisors in X_Σ to Y generate $\mathrm{Cl}(Y)$ we have to show that in the group generated by these divisors there are duals of the exceptional curves. This is equivalent to the fact that the system of equations given by the rows of U with the constant terms such that one is 1 and the remaining are 0 has an integral solution. And such solutions can be easily constructed using the methods and observations as in the proof of Lemma 3.6. $\square$

5. The spectrum of the Cox ring

The aim of this section is to prove Theorem 5.3 stating that the hypersurface $S \subset \mathbb{C}^{n+3}$ introduced in Construction 3.17 is the spectrum of the Cox ring of the minimal resolution X of a surface quotient singularity $\mathbb{C}^2/G$. Our argument is based on Theorem 6.4.3 and Corollary 6.4.4 in [**ADHL10**], which provide a characterization of the Cox rings via Geometric Invariant Theory. As before, we investigate S, its T-invariant open subset

$$W = S \setminus Z(\Sigma)$$

(independent of the choice of a simplicial fan Σ) and geometric quotient $Y = W/T$. We need to check a few properties of these spaces required by the assumptions of the results of [**ADHL10**]. Some of them are direct consequences ofthe observations we have already made. The remaining ones are proven in this section. It is worth noting that the quotients considered here are a special case of a much more general theory of good quotients of algebraic varieties by reductive group actions, developed by Białynicki-Birula and Święcicka in a series of papers including [**BBS96**], which can be useful for a possible generalization of our results.
The first property is the strong stability of the action of T on W – for the definition see [**ADHL10**, Def. 6.4.1]. Then, in Proposition 5.2, we prove the T-factoriality of S.

Proposition 5.1. *The action of T on W is strongly stable.*

Proof. The variety W from [**ADHL10**, Def. 6.4.1] becomes W defined above, and we take $W = W'$. Obviously, W' is T-invariant and the codimension of its complement in W is ≥ 2. Also, by Remark 4.1 all the orbits of T in W are closed. Finally, in Lemma 4.4 it is proven that T acts freely on W. □

Proposition 5.2. *S is T-factorial i.e. every T-invariant Weil divisor on S is principal.*

Proof. First notice that every T-invariant Weil divisor in S is a pull-back of a divisor in $Y = W/T$. This is because of dimension reasons: T is an n-dimensional torus acting freely on an $(n+2)$-dimensional variety W, and $S \setminus W$ is of codim ≥ 2 in S. Hence it is sufficient to show that the pull-backs of generators of $\mathrm{Pic}(Y)$ are principal. Their equations are $\{y_{i,j} = 0\}$ or $\{x_i = 0\}$. Thus the question is whether Cartier divisors defined by these functions are not multiples of Weyl divisors $\{y_{i,j} = 0\} \cap S$ and $\{x_i = 0\} \cap S$. We have to compute that valuations corresponding to local rings of S are 1 on $y_{i,j}$ and x_i.
The argument is the same for all functions x_i and $y_{i,j}$, so we may choose x_1 and check that it is not in the square of the maximal ideal of the localization of $\mathbb{C}[S]$ in a generic point of $\{x_1 = 0\} \cap S$. Since S is given by the equation

$$\sum_{i=1,2,3} y_{i,1}^{(\alpha_i)_1} \cdots y_{i,n_i}^{(\alpha_i)_{n_i}} \cdot x_i^{(\alpha_i)_{n_i+1}},$$

the ideal with respect to which we localize contains $y_{2,1}^{(\alpha_2)_1} \cdots y_{2,n_2}^{(\alpha_2)_{n_2}} x_2^{(\alpha_2)_{n_2+1}} + y_{3,1}^{(\alpha_3)_1} \cdots y_{3,n_3}^{(\alpha_3)_{n_3}} x_3^{(\alpha_3)_{n_3+1}}$. However, it is irreducible, so we cannot obtain from it any elements of the ideal dividing x_1, hence x_1 is a generator of the maximal ideal of the localization.

□

We are ready to complete the proof of the first of our main results.

THEOREM 5.3. *Let X be the minimal resolution of a surface quotient singularity. If S is as defined in Construction 3.17, then $S \simeq \mathrm{Spec}(\mathrm{Cox}(X))$.*

PROOF. S is a normal affine variety with the action of a torus T. We check that the conditions of Corollary 6.4.4 in [**ADHL10**] are satisfied.
First we observe that every invertible function on S is constant. Take any point

$$(v_0, u_1, v_{1,1}, \ldots, v_{1,n_1+1}, u_2, v_{2,1}, \ldots, v_{2,n_2+1}, u_3, v_{3,1}, \ldots, v_{3,n_3+1}) \in S.$$

Remember that variables $y_{1,1}$, $y_{2,1}$, $y_{3,1}$ appear in the equation of S with exponent 1 (see Remark 3.18). Thus the equations $x_i = u_i$, $y_{i,j} = v_{i,j}$ for all $i = 1, 2, 3$, $j = 0$ and $j > 1$ together with the equation of S determine a plane given by an equation of the form $q_1 y_{1,1} + q_2 y_{2,1} + q_3 y_{3,1} = 0$, passing through the chosen point. It is contained in S and has nonempty intersection with the affine space $y_{1,0} = y_{2,0} = y_{3,0} = 0$, also contained in S. As on an affine space all invertible functions are constant, restriction of such a function on S to both considered affine spaces is constant, so it is constant on S.
By Proposition 5.2 we know that S is T-factorial. Now $W \subset S$ is open and T-invariant subset such that $\mathrm{codim}_S(S \setminus W) \geq 2$. The action of T on W admits a good quotient, as it was observed in Remark 4.1. Finally, by Proposition 5.1 this action is strongly stable. Therefore it follows from [**ADHL10**, Cor. 6.4.4] that S is the spectrum of the Cox ring of X. □

6. Generators of the Cox ring

6.1. Generators. We construct a set of generators of $\mathrm{Cox}(X)$ where X is the minimal resolution of $\mathbb{C}^2/G$ for a small group $G \subset GL(2, \mathbb{C})$. To this end, we present $\mathrm{Cox}(X)$ as a subring of the coordinate ring of $\mathbb{C}^2/[G, G] \times T$. We expect that the ideas sketched here can be generalized and may form a basis for the extension of this work to higher dimensional quotient singularities, at least for some specific classes of groups. This set-up will be developed in a forthcoming paper [**DBW13**].
Our aim is to define a monomorphism

$$\phi \, : \, \mathrm{Cox}(X) \hookrightarrow \mathbb{C}[a, b]^{[G,G]}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}]$$

and describe the generators of $\phi(\mathrm{Cox}(X))$. Let the coordinate ring of $\mathbb{C}^{n+3}$ be

$$A = \mathbb{C}[y_0, y_{1,1}, \ldots, y_{1,n_1}, x_1, y_{2,1}, \ldots, y_{2,n_2}, x_2, y_{3,1}, \ldots, y_{3,n_3}, x_3].$$

We start from defining a homomorphism

$$\overline{\phi} \, : \, A \to \mathbb{C}[a, b]^{[G,G]}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}].$$

But first of all we prove a general statement about the decomposition of $\mathbb{C}[a, b]^{[G,G]}$ into the eigenspaces of the action of $Ab(G)$ and describe the Cox ring of a quotient singularity.

LEMMA 6.1. *Let V be an affine space and $\mathbb{C}[V]$ its coordinate ring. Then $\mathbb{C}[V]^{[G,G]}$ decomposes into $\mathbb{C}[V]^G$-modules of rank 1, associated to the characters of G:*

$$\mathbb{C}[V]^{[G,G]} = \bigoplus_{\mu \in G^\vee} \mathbb{C}[V]^G_\mu,$$

such that the action of G on $\mathbb{C}[V]^G_\mu$ is determined by the values of μ on G.

Proof. Look at $\mathbb{C}[V]^G \subset \mathbb{C}[V]^{[G,G]} \subset \mathbb{C}[V]$ and the inclusions of corresponding fields of fractions $\mathbb{C}(V)^G \subset \mathbb{C}(V)^{[G,G]} \subset \mathbb{C}(V)$. Note that $\mathbb{C}(V)^G$ means both the field of fractions of $\mathbb{C}[V]^G$ and the subfield of invariants of the induced action of G on $\mathbb{C}(V)$.

Consider $\mathbb{C}(V)$ as a Galois extension of $\mathbb{C}(V)^G$ with the Galois group G (see e.g. [**Ben93**, Prop. 1.1.1]). Then $\mathbb{C}(V)^{[G,G]}$ corresponds to a normal subgroup of G, so $\mathbb{C}(V)^G \subset \mathbb{C}(V)^{[G,G]}$ is also Galois with the Galois group $G/[G,G] = Ab(G)$. By the primitive element theorem this extension is generated by one element α, so $\mathbb{C}(V)^{[G,G]}$ is a vector space over $\mathbb{C}(V)^G$ spanned by $\{g(\alpha) \ : \ g \in Ab(G)\}$. This basis with $Ab(G)$ action is isomorphic to $Ab(G)$ acting on itself by multiplication, so $\mathbb{C}(V)^{[G,G]}$ is the regular representation of $Ab(G)$. Hence it splits into the sum of all irreducible representations of $Ab(G)$, which are one-dimensional as $Ab(G)$ is abelian, and each of them appears once in the decomposition (see [**FH91**, Cor. 2.18]):

$$\mathbb{C}(V)^{[G,G]} = \bigoplus_{\mu \in Ab(G)^\vee} \mathbb{C}(V)^G_\mu,$$

where $\mathbb{C}(V)^G_\mu = \{f \in \mathbb{C}(V)^{[G,G]} \ : \ g(f) = \mu(g)f\}$. Note that $G^\vee = Ab(G)^\vee \simeq Ab(G)$.

It remains to prove that $\mathbb{C}[V]^{[G,G]}$ is a direct sum of $\mathbb{C}[V]^G_\mu = \mathbb{C}(V)^G_\mu \cap \mathbb{C}[V]^{[G,G]}$. Every $f \in \mathbb{C}[V]^{[G,G]}$ can be written (uniquely) as a sum

$$f = \sum_{\mu \in Ab(G)^\vee} \frac{v_\mu}{w_\mu}$$

where $v_\mu/w_\mu \in \mathbb{C}(V)^G_\mu$. We have to show that $v_\mu/w_\mu \in \mathbb{C}[V]^{[G,G]}$. As the rows of the character table of an abelian group form a basis of a vector space, we can look at the image of f under such combination of elements of $Ab(G)$ that only the component corresponding to a chosen character will not vanish. Then on the right side we will have a chosen v_μ/w_μ multiplied by some complex coefficient and on the left side there will be a $\mathbb{C}$-linear combination of images of f under $Ab(G)$ action, so a polynomial. Thus indeed v_μ/w_μ is a polynomial. □

The following proposition shows that the homomorphism ϕ, which we are about to construct, describes a relation between the Cox rings of a (2-dimensional quotient) singularity and its (minimal) resolution.

Proposition 6.2. *For a complex vector space V with an action of $G \subset GL(V,\mathbb{C})$ we have*

$$\mathrm{Cox}(V/G) \simeq \mathbb{C}[V]^{[G,G]}.$$

Proof. Rank one $\mathbb{C}[V]^G$-modules $\mathbb{C}[V]^G_\mu$ in the decomposition of $\mathbb{C}[V]^{[G,G]}$ in Lemma 6.1 can be identified with $\mathcal{O}(V/G)$-modules of global sections of $\mathcal{O}_{V/G}(D)$ for $D \in \mathrm{Cl}(V/G) \simeq Ab(G)$. □

Notation 6.3. By $\sigma_i(a,b)$ for $i = 1,2,3$ we denote homogeneous polynomials invariant under the action of $[G,G]$ on $\mathbb{C}[a,b]$ which are eigenvectors of the action of $Ab(G)$ on $\mathbb{C}[a,b]^{[G,G]}$, chosen such that the set $\{\sigma_1(a,b), \sigma_2(a,b), \sigma_3(a,b)\}$ generates $\mathbb{C}[a,b]^{[G,G]}$ (as a $\mathbb{C}$-algebra).

Remark 6.4. For all small subgroups $G \subset Gl(2,\mathbb{C})$ such polynomials $\sigma_i(a,b)$ can be found and are determined uniquely up to constants. The latter follows by analyzing the numbers of $[G,G]$-invariants in small gradations (obtained e.g. from Molien

series, which can be computed in [**GAP12**]). In Example 6.11 we give a direct proof of existence of such generating sets, i.e. we write them down. In most of the cases considered here the homogeneity condition is forced by the assumption that a polynomial is an eigenvector of $Ab(G)$. However, sometimes it is not – for example, $Ab(BI)$ is trivial, so all invariants are the eigenvectors, but only the choice of homogeneous polynomials gives the right result.

As before, $\chi_i(t_0, \ldots, t_{n-1})$ denotes the monomial with exponents given by the i-th column of U, i.e. one of the characters on the Picard torus T used to define the action of T on $\mathbb{C}^{n+3}$.

DEFINITION 6.5. Define $\overline{\phi}$ as follows:

$$\begin{aligned}\overline{\phi}(x_i) &= \sigma_i(a,b)\chi_{k(i)}(t_0, \ldots, t_{n-1}),\\ \overline{\phi}(y_0) &= \chi_0(t_0, \ldots, t_{n-1}),\\ \overline{\phi}(y_{i,j}) &= \chi_{k(i,j)}(t_0, \ldots, t_{n-1}),\end{aligned}$$

where $k(i)$ and $k(i,j)$ are the numbers of the column of U (counting from 0) corresponding to the variable x_i and $y_{i,j}$ respectively (ordered as in the definition of A above). To be precise, $k(i) = i + \sum_{p=1}^{i} n_i$ and $k(i,j) = i + j - 1 + \sum_{p=1}^{i-1} n_i$.

LEMMA 6.6. *The homomorphism $\overline{\phi}$ factors through*

$$\phi\ :\ \mathrm{Cox}(X) \to \mathbb{C}[a,b]^{[G,G]}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}].$$

PROOF. We show that the image under $\overline{\phi}$ of the equation of $S = \mathrm{Spec}(\mathrm{Cox}(X))$, described in Construction 3.17, is zero. This equation is the sum of three monomials corresponding to the branches and their vectors of exponents are α_i, which are in the kernel of A_i. Thus the image of the monomial corresponding to the i-th branch is $t_0 \cdot \sigma_i(a,b)^{(\alpha_i)_{n_i+1}}$. Hence it is sufficient to argue that

$$\sigma_1(a,b)^{(\alpha_1)_{n_1+1}} + \sigma_2(a,b)^{(\alpha_2)_{n_2+1}} + \sigma_3(a,b)^{(\alpha_3)_{n_3+1}} = 0.$$

From Lemma 3.12 we know that $(\alpha_i)_{n_i+1} = p_i$, i.e. the numerator of the fraction describing the i-th branch of the resolution diagram. These numbers can be read out from the table in [**Bri68**, Satz 2.11]. On the other hand, Lemma 2.2 gives an explicit description of the commutator subgroup of each considered group G. The exponents in the equation of the Du Val singularity $\mathbb{C}^2/[G,G]$ are given e.g. in [**Rei**]. Compare them to the numbers p_i – they are the same. Hence it is enough to check that $\sigma_1(a,b), \sigma_2(a,b), \sigma_3(a,b)$ satisfy the relation in $\mathbb{C}[a,b]^{[G,G]}$ (up to multiplication by some constants). This can be done in a straightforward way, as they are listed in Example 6.11. □

LEMMA 6.7. *The homomorphism ϕ is a monomoprhism.*

PROOF. Assume that a polynomial $w \in A$ is in $\ker \overline{\phi}$. Subtracting some multiple of the generator of $I(S)$ and multiplying w by some $v \in A$, which does not contain x_3, we get w' such that its degree as a polynomial of one variable x_3 is smaller than $(\alpha_3)_{n_3+1}$. Thus when we substitute 1 for all t_i in $\overline{\phi}(w')$ we obtain a polynomial in variables a, b which is not divisible by the relation between σ_1, σ_2 and σ_3. However, $\overline{\phi}(w') = \overline{\phi}(wv) = 0$, so if we consider again w' as a polynomial of x_3, the polynomials of x_1, x_2, y_0, $y_{i,j}$ which are its coefficients are mapped by $\overline{\phi}$ to 0. As σ_1, σ_2 and these characters $\chi_i(t_0, \ldots, t_{n-1})$ which do not correspond

to the variables x_i are independent, $w' = 0$. Hence $wv \in I(S)$ and finally, because $I(S)$ is prime and $v \notin I(S)$, we obtain $w \in I(S)$. $\square$

REMARK 6.8. Define an action of $Ab(G)$ on $\mathbb{C}[a,b]^{[G,G]}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}]$ as follows:

- the action on σ_i is induced from the action of G on $\mathbb{C}[a,b]$ (which, by definition of σ_i, is multiplication by $c_i \in \mathbb{C}$,
- the action on the variable $t_{k(i)}$, which is the character of T corresponding to x_i, is multiplication by $1/c_i$ (so $\phi(x_i)$ are fixed points),
- we require that the remaining characters of T are fixed (with this assumption the action on the coordinates of T is uniquely determined).

Then the image of ϕ is just the invariant ring of this action.

As a result of Lemmata 6.6 and 6.7 we obtain

THEOREM 6.9. $\mathrm{Cox}(X) \subset \mathbb{C}[a,b]^{[G,G]}[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}]$ *is generated by the images of the variables under* ϕ, *as listed in Definition 6.5, i.e.*

- $\sigma_i(a,b) \cdot \chi_{k(i)}(t_0, \ldots, t_{n-1})$ *for* $i = 1,2,3$ *and*
- $\chi_0(t_0, \ldots, t_{n-1})$ *and* $\chi_{k_{i,j}}(t_0, \ldots, t_{n-1})$ *for* $i = 1,2,3,\ j = 1, \ldots, n_i$.

Now we use Proposition 6.2 to identify $\mathbb{C}[a,b]^{[G,G]}$ with $\mathrm{Cox}(\mathbb{C}^2/G)$ and we look at the composition

$$\mathrm{Cox}(X) \xrightarrow{\phi} \mathrm{Cox}(\mathbb{C}^2/G)[t_0^{\pm 1}, t_1^{\pm 1}, \ldots, t_{n-1}^{\pm 1}] \xrightarrow{t_0, \ldots, t_{n-1} \mapsto 1} \mathrm{Cox}(\mathbb{C}^2/G).$$

We may think of two kinds of generators of $\mathrm{Cox}(X)$. Some are pull-backs of the generators of $\mathrm{Cox}(\mathbb{C}^2/G)$, which come from the $Ab(G)$ action eigenspaces in $\mathbb{C}[a,b]^{[G,G]}$, (these are $\sigma_i \cdot \chi_{k(i)}$) and other are mapped to $1 \in \mathrm{Cox}(\mathbb{C}^2/G)$. The first kind depends on the group structure and the second one contains the information on the intersection numbers of components in the special fibre. This idea of describing the generators of $\mathrm{Cox}(X)$ seems to be more general than just the two-dimensional case. We finish with a few words about the geometric meaning of these results. The dual map to ϕ is just the morphism from the torus bundle $\mathbb{C}^2/[G,G] \times T$ to $\mathrm{Spec}(\mathrm{Cox}(X))$. As ϕ is a monomorphism, the dual is a dominant map. And from Remark 6.8 it follows that this is the quotient by an action of $Ab(G)$.

6.2. Examples. The examples below describe the generators of $\mathrm{Cox}(X)$ or the homomorphism ϕ explicitly in a few interesting cases. Also, in Example 6.11, we find the eigenvectors of $Ab(G)$ which generate $\mathbb{C}[a,b]^{[G,G]}$ for all small groups $G \subset GL(2,\mathbb{C})$.

EXAMPLE 6.10 (Binary dihedral groups BD_{4n}). The commutator subgroup of BD_{4n} is $\mathbb{Z}_n = \langle diag(\varepsilon_n, \varepsilon_n^{-1}) \rangle$. The ring of invariants of the action of $[BD_{4n}, BD_{4n}]$ on $\mathbb{C}[x,y]$ is generated by xy, x^n and y^n. However, only the first monomial is an eigenvector of the action of $Ab(BD_{4n})$ on this ring of invariants and we have to find suitable linear combinations of the remaining two. As before the coordinates on $\mathbb{C}^2 \times (\mathbb{C}^*)^n$ are $(a, b, t_0, \ldots, t_{n-1})$.

If n is even then the generators of $\mathrm{Cox}(X)$ are

$$\begin{aligned} \phi(x_j) &: \quad i(a^n + b^n)t_1, (a^n - b^n)t_2, 2^{\frac{2}{n}} abt_{n-1}, \\ \phi(y_0), \phi(y_{i,j}) &: \quad \frac{t_1 t_2 t_3}{t_0^2}, \frac{t_0}{t_1^2}, \frac{t_0}{t_2^2}, \frac{t_0 t_4}{t_3^2}, \frac{t_3 t_5}{t_4^2}, \frac{t_4 t_6}{t_5^2}, \ldots, \frac{t_i t_{i+2}}{t_{i+1}^2}, \ldots, \frac{t_{n-3} t_{n-1}}{t_{n-2}^2}, \frac{t_{n-2}}{t_{n-1}^2}. \end{aligned}$$

And if n is odd, we have

$$\begin{aligned} \phi(x_j) &: \quad (-ia^n+b^n)t_1, (a^n-ib^n)t_2, 2^{\frac{2}{n}}abt_{n-1}, \\ \phi(y_0), \phi(y_{i,j}) &: \quad \frac{t_1t_2t_3}{t_0^2}, \frac{t_0}{t_1^2}, \frac{t_0}{t_2^2}, \frac{t_0t_4}{t_3^2}, \frac{t_3t_5}{t_4^2}, \frac{t_4t_6}{t_5^2}, \dots, \frac{t_it_{i+2}}{t_{i+1}^2}, \dots, \frac{t_{n-3}t_{n-1}}{t_{n-2}^2}, \frac{t_{n-2}}{t_{n-1}^2}. \end{aligned}$$

We can use the formula for ϕ (more precisely, for the associated morphism of varieties) in the case of odd n to correct a false statement on page 9 of [**FGAL11**]. The authors describe the quotient of $\mathbb{C}^{n+3}$ by the Picard torus action as

$$\begin{aligned} V = \{Z_2^4 - Z_5Z_6 = Z_1Z_2^2 - Z_3Z_4 = Z_2^2Z_4 - Z_3Z_6 = \\ = Z_2^2Z_3 - Z_4Z_5 = Z_4^2 - Z_1Z_6 = Z_3^2 - Z_1Z_5 = 0\} \end{aligned}$$

and attempt to realize $\mathbb{C}^2/BD_{4n}$ as a subvariety of V. They suggest that it is isomorphic to

$$V' = V \cap \{Z_1^k + Z_3 + Z_4 = 0\},$$

where $k = (n-1)/2$. However, this variety is reducible. One component (of dimension 2) is given by $Z_1 = Z_3 = Z_4 = Z_2^4 - Z_5Z_6 = 0$ and the second one, isomorphic to $\mathbb{C}^2/BD_{4n}$, is the closure of the set of points of V' with at least one of Z_1, Z_3, Z_4 nonzero.

To obtain the full set of equations of the second component we first apply the quotient morphism described in [**FGAL11**, Lemma 4.2] to the image of ϕ, i.e. we compute monomials $Z_1, \dots, Z_6$.

The relations between these monomials for some small values of n can be computed for example in Singular, [**DGPS12**]. Thus we find two more equations, namely

$$Z_1^{k-1}Z_3 + Z_2^2 + Z_5 = 0 \quad \text{and} \quad Z_1^{k-1}Z_4 + Z_2^2 + Z_6 = 0.$$

It turns out that they are sufficient for all odd n. i.e.

$$V \cap \{Z_1^k + Z_3 + Z_4 = Z_1^{k-1}Z_3 + Z_2^2 + Z_5 = Z_1^{k-1}Z_4 + Z_2^2 + Z_6 = 0\}$$

is irreducible and by a direct computation one can check that its coordinate ring is isomorphic to the one of $\mathbb{C}^2/BD_{4n}$.

This observation does not change anything in the main results of [**FGAL11**]. However, it is a convincing example that the ideas used there may be hard to generalize to more complicated singularities.

Example 6.11. Let G be a finite nonabelian small subgroup of $GL(2,\mathbb{C})$. We compute the eigenvectors of the induced action of $Ab(G)$ which generate $\mathbb{C}[x,y]^{[G,G]}$. We use the list of generators of $[G,G]$-invariants from [**DZ93**] and Corollary 2.3. This data together with the description of the special fibre of the minimal resolution of $\mathbb{C}^2/G$ (in section 2.2) is sufficient to write down ϕ in all considered cases.

(1) For $G = BD_{n,m}$ we have $[G,G] \simeq \mathbb{Z}_n \subset SL(2,\mathbb{C})$. The invariants of $[G,G]$ are generated by

$$xy, \quad x^n, \quad y^n$$

with the relation $(xy)^n - x^ny^n = 0$. Invariants that are eigenvectors of $Ab(G)$ are

$$xy, \quad x^n + y^n, \quad x^n - y^n$$

for even n and

$$xy, \quad x^n + iy^n, \quad x^n - iy^n$$

for odd n.

(2) For $G = BT_m$ there is $[G,G] = BD_2$ and its invariants are generated by

$$x^2y^2, \quad x^4+y^4, \quad xy(x^4-y^4)$$

with the relation $-4(x^2y^2)^3 + (x^2y^2)(x^4+y^4)^2 - (xy(x^4-y^4))^2 = 0$. The last polynomial is an invariant of BT and because $AB(BT_m)$ is generated by an element of BT multiplied by a root of unity, it is an invariant of $AB(BT_m)$. The remaining eigenvectors of $Ab(G)$ are

$$x^4+y^4+2i\sqrt{3}x^2y^2 \quad \text{and} \quad x^4+y^4-2i\sqrt{3}x^2y^2.$$

(3) For $G = BO_m$ the invariants of $[G,G] = BT$ are generated by

$$A = \sqrt[4]{108}xy(x^4-y^4),$$

$$B = -(x^8+14x^4y^4+y^8),$$

$$C = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$$

with the relation $A^4+B^3+C^2 = 0$. Moreover, these generators span the eigenspaces.

(4) Finally, for $G = BI_m$ the invariants of $[G,G] = BI$ are generated by

$$D = \sqrt[5]{1728}xy(x^{10}+11x^5y^5-y^{10}),$$

$$E = -(x^{20}+y^{20}) + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10},$$

$$F = x^{30}+y^{30}+522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10}+x^{10}y^{20})$$

with the relation $D^5+E^3+F^2 = 0$. As before, these generators span the eigenspaces.

Example 6.12. There is a case where the morphism of varieties induced by ϕ is an embedding of trivial torus bundle in $\mathrm{Spec}(\mathrm{Cox}(X))$: the Du Val singularity E_8. This is because the commutator subgroup BI is the whole group and the abelianization is trivial. By Remark 6.8 ϕ induces a morphism from $\mathbb{C}^2/[BI,BI] \times (\mathbb{C}^*)^8$ to $\mathrm{Spec}(\mathrm{Cox}(X))$ which is a quotient by the trivial group action, so the image is isomorphic to $\mathbb{C}^2/BI \times (\mathbb{C}^*)^8$.

Example 6.13. Let us write down the generators of $\mathrm{Cox}(X)$ in a chosen case which is not a Du Val singularity: $G = BD_{23,39}$. It was already explored in Examples 2.6 and 3.20, where the fibre diagram and the extended intersection matrix are shown. As before, choose the coordinates on $\mathbb{C}^{10}$ to be

$$(y_0, y_{1,1}, x_1, y_{2,1}, x_2, y_{3,1}, y_{3,2}, y_{3,3}, y_{3,4}, x_3).$$

$[BD_{23,39}, BD_{23,39}] \simeq \mathbb{Z}_{23} \subset SL(2,\mathbb{C})$, n is odd, so the generators are

$$\phi(x_1) = (-ia^{23}+b^{23})t_1, \quad \phi(x_2) = (a^{23}-ib^{23})t_2, \quad \phi(x_3) = -i\sqrt[23]{4}abt_6,$$
$$\phi(y_0) = t_1t_2t_3t_0^{-3}, \quad \phi(y_{1,1}) = t_0t_1^{-2}, \quad \phi(y_{2,1}) = t_0t_2^{-2},$$
$$\phi(y_{3,1}) = t_0t_4t_3^{-4}, \quad \phi(y_{3,2}) = t_3t_5t_4^{-2}, \quad \phi(y_{3,3}) = t_4t_6t_5^{-2}, \quad \phi(y_{3,4}) = t_5t_6^{-3}.$$

References

[ADHL10] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, *Cox Rings*, arXiv:1003.4229 [math.AG] (2010).

[AW10] Marco Andreatta and Jarosław A. Wiśniewski, *On the Kummer Construction*, Revista Matematica Complutense **23** (2010), 191–251.

[BBS96] Andrzej Białynicki-Birula and Joanna Święcicka, *Open subset of projective spaces with a good quotient by an action of a reductive group*, Transformation Groups **1** (1996), 153–185.

[Ben93] David J. Benson, *Polynomial invariants of finite groups*, LMS Lecture Notes Series, vol. 190, Cambridge University Press, 1993.

[Bri68] Egbert Brieskorn, *Rationale Singularitäten komplexer Flächen*, Inventiones Mathematicae **4** (1968), 336–358.

[CLS11] David Cox, John Little, and Hal Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, 2011.

[DBW13] Maria Donten-Bury and Jarosław A. Wiśniewski, *Total coordinate rings of resolutions of a quotient singularity.*

[DGPS12] Wolfram Decker, Gert-Martin Greuel, Gerhard Pfister, and Hans Schönemann, SINGULAR *3-1-5 — A computer algebra system for polynomial computations*, http://www.singular.uni-kl.de.

[Don11] Maria Donten, *On Kummer 3-folds*, Revista Matematica Complutense **24** (2011), 465–492.

[DZ93] Dennis Deturck and Wolfgang Ziller, *Spherical minimal immersions of spherical space forms*, Proceedings in Symposia in Pure Mathematics **54** (1993), 111–120.

[FGAL11] Laura Facchini, Víctor González-Alonso, and Michał Lasoń, *Cox rings of Du Val singularities*, Le Matematiche **66** (2011), 115–136.

[FH91] William Fulton and Joe Harris, *Representation Theory, A First Course*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, 1991.

[GAP12] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.5.6*, 2012.

[HS10] Jürgen Hausen and Hendrik Süß, *The cox ring of an algebraic variety with torus action*, Advances in Mathematics **225** (2010), no. 2, 977–1012.

[Kol93] János Kollár, *Shafarevich maps and plurigenera of algebraic varieties*, Invent. Math. **113** (1993), 177–215.

[Lun73] Domingo Luna, *Slices étales*, Bull. Soc. Math. de France **33** (1973), 81–105.

[LV09] Antonio Laface and Mauricio Velasco, *A survey on Cox rings*, Geometriae Dedicata **139** (2009), 269–287.

[Rei] Miles Reid, *The Du Val singularities* A_n, D_n, E_6, E_7, E_8, http://homepages.warwick.ac.uk/˜masda/surf.

[Rei97] ———, *Surface cyclic quotient singularities and Hirzebruch-Jung resolutions*, http://homepages.warwick.ac.uk/˜masda/surf/, 1997.

[Rie77] Oswald Riemenschneider, *Die Invarianten der endlichen Untergruppen von GL(2,C)*, Mathematische Zeitschrift **153** (1977), 37–50.

[Stu93] Bernd Sturmfels, *Algorithms in Invariant Theory*, Springer-Verlag, 1993.

[Sum74] Hideyasu Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.

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